JIS GROUP

ONLINE COURSE WARE

SUBJECT NAME: MATHEMATICS III

SUBJECT CODE: M 301

TOTAL NO. OF LECTURES: 44

CONTACT HOURS: 44 HOURS.

CREDIT: 4

Target Streams: ECE, AEIE, EE, ME, CE

Module No	Lecture No	Topic	Application	Reference Book
MODULE I: Fourier Series and Fourier Transform	1	Introduction, Periodic functions: Properties, Even & Odd functions: Properties,	**1.** In Electrical engineering , the fourier transform is used to analyze varying voltages and currents **2.** Fourier transform is used in the spectrum analysis of signals. Some application of this are in communication systems, image and video processing, Biomedical engineering. Oil extraction and power quality analysis.	1. Dutta: A Textbook of Engineering Mathematics Vol.1 & 2, New Age International. 2. Lakshminarayan- Engineering Math 3. 3. Grewal B S: Higher Engineering Mathematics (thirtyfifthedn) - Khanna Pub. 4. Sarveswarao: Engineering Mathematics, Universities Press 5. Jana- Undergradute Mathematics. 6. Kreyzig E: Advanced Engineering Mathematics - John Wiley and Sons.
	2	Special wave forms: Square wave, Half wave Rectifier, Full wave Rectifier, Saw-toothed wave, Triangular wave. Euler's Formulae for Fourier Series,		
	3	Fourier Series for functions of period 2π, Fourier Series for functions of period , Dirichlet's conditions, Sum of Fourier series. Examples. Theorem for the convergence of Fourier Series		
	4	Fourier Series of a function with its periodic extension. Half Range Fourier Series: Construction of Half range Sine Series,		
	5	Construction of Half range Cosine Series. Parseval's identity (statement only).Examples		
	6	Fourier Integral Theorem, Fourier Transform of a function, Fourier Sine and Cosine Integral Theorem with examples. Fourier Cosine & Sine Transforms.		
	7	Transform of some standard function, Fourier, Fourier Cosine & Sine Transforms of elementary		

		functions, Properties of Fourier Transform. : Linearity, Shifting, Change of scale, Modulation		
	8	Fourier Transform of Derivatives, Examples. Convolution Theorem with examples.		
	9	Inverse Fourier transform or Fourier Integral, Fourier Sine and Cosine Integral Theorem Inverse Property of Inverse Fourier Transform.		
	10	Some examples with solution of Fourier Transform, Exercise.		
MODULE II: Probability Distributions	11	Definition of random variable. Continuous and discrete random variables. probability mass function for single variable only	1. Probability theory has huge application in everyday life in risk assessment and modeling. 2. Governments apply probabilistic methods in environmental regulation, entitlement analysis. 3. Probability theory also has huge application in Engineering. In Network Analysis (CSE, IT, ECE), network contains nodes which can be deterministic or Probabilistic . 4. In reliability (EIE), probability of failure and probability of availability are two of the most important aspects	**1.** Gupta S. C and Kapoor V K: Fundamentals of Mathematical Statistics - Sultan Chand & Sons. **2.** Lipschutz S: Theory and Problems of Probability (Schaum's Outline Series) - McGraw Hill Book. Co. **3.** Spiegel M R: Theory and Problems of Probability and Statistics (Schaum's Outline Series) - McGraw Hill Book Co. **4.** Goon A.M., Gupta M K and Dasgupta B: Fundamental of Statistics - The World Press Pvt. Ltd. 5. Delampady, M:
	12	Distribution Function, Probability density function for single variable only.		
	13	Determination of Mean, Variance and standard deviation of discrete and continuous distribution		
	14	Some important discrete distributions: Binomial, Determination of Mean, Variance and standard deviation .		
	15	Poisson distributions: Determination of Mean,		

		Variance and standard deviation.	of safely critical systems like aircraft, nuclear power plant etc. In semi conductor physics (EIE,ECE), Fermi-Dirac function or Fermi-Dirac probability function is extremely important for understanding the operation of semiconductor devices like diodes, transistors etc. It is also important to understand the operations of LED and semi conductor laser	Probability & Statistics, Universities Press 6. Bhat: Modern Probability Theory, New Age International
	16	Continuous distributions: Normal. Determination of Mean, Variance and standard deviation.		
	17	Problems related to Standard Normal distribution		
	18	Correlation &Regression analysis		
	19	Curve fitting		
	20	Least Square method		
MODULE III: Calculus of Complex Variable	21	Introduction to Functions of a Complex Variable, Concept of Limit, Continuity	1.The treatment of resistors, capacitors, and inductors can then be unified by introducing imaginary, frequency dependent resistances for the latter two and combining all three in a single complex number called the impedance. 2. Electrical engineers and some physicists use the	1. Spiegel M R: Theory and Problems of Complex Variables (Schaum's Outline Series) - McGraw Hill Book Co. 2. Chowdhury: Elements of Complex Analysis, New Age International 3. Grewal B S: Higher Engineering Mathematics (thirtyfifthedn) - Khanna Pub. 4. Sarveswarao: Engineering Mathematics, Universities Press 5. Jana- Undergradute Mathematics.
	22	Concept of Differentiability. Analytic functions Cauchy-Riemann Equations (statement only). Sufficient condition for a function to be analytic		
	23	Laplace Equation Harmonic function and Conjugate Harmonic function, related problems. Construction of		

		Analytic functions. Milne Thomson method, related problems	letter j for imaginary unit since i is typically reserved for varying currents and may come into conflict with i. This approach is called phaser calculus. This use is also extended into digital signal processing and digital image processing which utilize digital version of fourier analysis (and wavelet analysis) to transmit ,compress, restore and otherwise process digital audio signals, still images and video signals. 3. Complex numbers are used in signal analysis and other fields for a convenient description for periodically varying signals . For given real functions representing , actual physical quantities, often in terms of sines and cosines corresponding	6. Kreyzig E: Advanced Engineering Mathematics - John Wiley and Sons. **7.** Dutta: A Textbook of Engineering Mathematics Vol.1 & 2, New Age International. 8. Lakshminarayan- Engineering Math 3.
	24	Introduction to Complex Integration Concept of simple curve, closed curve, smooth curve & contour. Cauchy's theorem (statement only).		
	25	Cauchy-Goursat theorem (statement only). Line integrals along a piecewise smooth curve. Examples		
	26	Cauchy's integral formula		
	27	Cauchy's integral formula for the derivative of an analytic function, Cauchy's integral formula for the successive derivatives of an analytic function. Examples		
	28	Taylor's series, Laurent's series. Examples		
	29	Zero of an Analytic function, order of zero, Singularities of an analytic function. Isolated and non-isolated singularity, essential		

		singularities. Poles: simple pole, pole of order m	complex functions are considered of which the real parts are the original quantities. 4. Complex numbers are used a great deal in electronics. The main reason for this is they make the whole topic of analyzing and understanding alternating signals much easier. 5. It helps us to understand the behavior of circuits which contains reactance when we apply a.c signals . It gives us a new way to think about oscillations. This is useful when we want to apply the concepts like the conservation of energy to understand the behavior of systems which range from simple mechanical pendulum to a quartz crystal oscillator. 6. In order to	
	30	Examples on determination of singularities and their nature. Residue, Cauchy's Residue theorem (statement only), problems on finding the residue of a given function		
	31	Introduction Conformal transformation		
	32	Bilinear transformation, simple problems		

			analyze A.C circuits it became necessary to represent multi - dimensional quantities. In order to accomplish this task, scalar numbers were abandoned and complex numbers were used to express the two dimensions of frequency and phase shift at one time.	
MODULE IV: Partial Differential Equation (PDE) & Series Solution Of Ordinary Differential Equation (ODE)	33	Basic Concepts of PDE	1. Partial differential Equations forms an essential part of engineering mathematics. The origin and application of such equations occur in a variety of different fields ranging from fluid dynamics , electromagnetism, heat conduction and diffusion to quantum mechanics , wave propagation and general relativity. 2. The differential	1. Bronson R: Differential Equations (Schaum's Outline Series) - McGraw Hill Book Co. **2.** Ross S L: Differential Equations - John Willey & Sons. **3.** Sneddon I. N.: Elements of Partial Differential Equations 4. Grewal B S: Higher Engineering Mathematics (thirtyfifthedn) - Khanna Pub. **5.**. Kreyzig E: Advanced Engineering Mathematics
	34	Solution PDEs		
	35	Solution by the method of Separation of Variables: One-dimensional Wave Equation		
	36	One-dimensional Heat Equation		
	37	Introduction to Series Solution		
	38	Example on Power series and Legendre Function		
	39	Rodrigue’s Formula		
	40	Recurrence Relations on Legendre Polynomials		
	41	Orthogonality of Legendre Polynomials		
	42	Bessel’s Function		

	43	Recurrence relations on Bessel's Function	equation representing the signal helps in the calculation of different transforms like Fourier , Z-Transform, etc. This transform is done by the spectrum analyzer. Also the differential equation used to find maxima and minima of the quantity. 3. The differential equations often model real life problems. They include but are not limited to the following: - The rate of water flowing out of a container - The position of an object as it oscillates to equilibrium - The state of a system over time 4. Electrical engineering deals with the manipulation of electrons and photons to produce products that benefit humanity.	- John Wiley and Sons. 6. Prasad: Partial Differential Equations, New Age International
	44	Elementary Bessel's Function		

			5. The design of these products is based on scientific principles and theories that are best described mathematically. Mathematics is thus the universal language of electrical engineering science. 6. Digital signal processing is an important area within electrical engineering. The digitization, modulation, transmission, demodulation, and reception of signals is vital to modern communications. 7. Image pro-cessing and pattern recognition techniques fall within the purview of digital signal processing. This is the area which requires pre notion or pre requisites of ordinary differential equation.	

MODULE I

Fourier Series and Fourier Transform

Lecture 1:

Introduction to Fourier Series

There are many types of series expansions for functions. The Maclaurin series, Taylor series, Laurent series are some such expansions. But these expansions become valid under certain strong assumptions on the functions (those assumptions ensure convergence of the series). Fourier series also express a function as a series and the conditions required are fairly good and suitable when we deal with signals.
Suppose f is a real valued function from R to R. In this note, we deal with the following three questions:

- When does f has a Fourier seriesexpansion?
- How we find the expansion?
- What are the main properties of thisexpansion?

Existence of a Fourier series expansion:

There are three conditions which guarantees the existence of valid Fourier expansion for a given function. These conditions are collectively called **Dirichlet conditions:**

1. f is a periodic function on R.This means that there exists a period $T\geq 0$ such that

$$f(x)=f(x+T) \text{ for all } x \in \mathbb{R}.$$

2. f has only a finite number of maxima and minima in aperiod.

3 f has atmost a finite number of discontinuous points inside aperiod.

4 f is integrable over the period of the function.

It should be noted that the second and third conditions are satisfied by many real valued functions that we deal with, inside any finite interval. But periodicity is a condition that is satisfied by very few functions,for example, *constant function*, *sine, cos, tan* and their combinations. But we can consider any function defined on a finite interval $[a,b]$ (or(a,b)) as a periodic function on R by thinking that the function is extended to R by repeating the values in$[a,b]$to the remaining part of R.

Even & Odd functions and their Properties:

A function f(x)is said to be even function if f(-x)=f(x) for all values of x; e.g. the functions cos(x), x^2 all are even functions.

A function f(x)is said to be an odd function if f(-x)=-f(x) for all values of x; e.g. the functions sin(x), x^3 all are odd functions.

Graph of an even function: One of the most important properties of even functions is

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Graph of an odd function: One of the most important properties of even functions is

$$\int_{-a}^{a} f(x)dx = 0$$

Lecture 2:

Special wave forms:

The graph of every periodic functions runs like a wave-this is wave-form. Below we show some typical wave-form which are usually met in communication engineering:

(i) Square waveform: Consider the periodic function f(x) defined by

$$f(x)=-k, \quad -a<x<0$$
$$=k, \quad 0<x \le a$$

And f(x+2a)=f(x) for all x

This kind of graph is known as Square Waveform.

(ii)Half wave Rectifier: Consider the periodic function f(x) defined by

$$f(x)=-k\sin x, \quad 0\le x\le \pi$$
$$=0, \quad \pi \le x \le 2\pi$$

And $f(x+2\pi)=f(x)$ for all x

This kind of graph is known as Half wave Rectifier.

(iii) Full wave Rectifier: Consider the periodic function f(x) defined by

$f(x)=-k\sin x, \quad 0 \le x \le \pi$

And $f(x+\pi)=f(x)$ for all values of x

This kind of graph is known as Full wave Rectifier.

(iv) Saw-toothed wave: Consider the periodic function f(x) defined by

$f(x)=x, \quad -a \le x \le a$

And $f(x+2a)=f(x)$ for all values of x

This kind of graph is known as Saw-toothed Waveform.

(v) Triangular wave: Consider the periodic function f(x) defined by

$$f(x) = 1 + \frac{2x}{a}, \quad -a < x \le 0$$

$$= 1 - \frac{2x}{a}, \quad 0 \le x \le a$$

And $f(x+2a)=f(x)$ for all values of x

This kind of graph is known as Triangular Waveform.

Examples:

1. Extend the function $f(x)=0, \ -3<x<0$

$\qquad = x^2, \ 0<x<3$

to a periodic function.

Sol. This function is defined on the interval (-3, 3) only. To extend in periodic form just define the function on $(-\infty,\infty)$ by the rule $f(x+6)=f(x)$ for all values of x.

<u>Euler's Formulae for Fourier Series:</u>

Let f(x) be defined and integrable in the interval (-T,T).

Extend the function to a periodic function of period 2T by defining $f(x+2T)=f(x)$ for all values of x of period 2T. The Fourier series of f(x) is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right)$$

Where a_n, b_n are called Fourier co-efficient and these are, according to Euler,

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx$$

Where n=1,2,3,…….

1. **Examples:** Consider the function f(x)=3, $0<x\le 5$
   $=-3,\ -5<x\le 0$

We extend the function by defining f(x+10)=f(x) for all x. so this becomes a periodic function of period 10. This gives a square waveform.

Sol. The Fourier co-efficient, according to Euler Formula, are

$$a_0 = \frac{1}{5}\int_{-5}^{5} f(x)dx$$

$$= \frac{1}{5}\left\{-3\int_{-5}^{0} dx + 3\int_{0}^{5} dx\right\} = 0$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\left\{-3\int_{-5}^{0} \cos\frac{n\pi x}{5}dx + \int_{0}^{5} f(x)\cos\frac{n\pi x}{5}dx\right\}$$

$$= \frac{1}{5}\left\{-3\int_{0}^{5} \cos\frac{n\pi x}{5}dx + \int_{0}^{5} f(x)\cos\frac{n\pi x}{5}dx\right\} = 0$$

And $b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}dx$

$$= \frac{2}{5}\int_{0}^{5} f(x)\sin\frac{n\pi x}{5}dx$$

$$= \frac{6}{5}\int_{0}^{5} \sin\frac{n\pi x}{5}dx$$

$$= -\frac{6}{5}\left[\frac{\cos\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right]_0^5 = -\frac{6}{n\pi}(\cos n\pi - 1) = \frac{6(1-\cos n\pi)}{n\pi}$$

Therefore the Fourier series of f(x) is

$$\frac{0}{2}+\sum_{n=1}^{\infty}(0.\cos\frac{n\pi x}{5}+\frac{6(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{T})$$

$$=\sum_{n=1}^{\infty}\frac{6}{\pi}.\frac{(1-\cos n\pi)}{n}\sin\frac{n\pi x}{5}$$

$$=\frac{6}{\pi}\left\{(1-\cos\pi)\sin\frac{\pi x}{5}+\frac{(1-\cos 2\pi)}{2}\sin\frac{2\pi x}{5}+\frac{(1-\cos 3\pi)}{3}\sin\frac{3\pi x}{5}+..........\right\}$$

We see f(0)=-3 but the values of the Fourier series at x=0 is $\frac{6}{\pi}\{0+0+0+......\}=0$

Lecture 3:

Fourier Series for functions of period 2π:

The above Fourier series for T= π i.e. the Fourier series for the function f(x) defined and integrable on $(-\pi,\pi)$ and $f(x+2\pi)=f(x)$ for all values of x is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Where the Fourier co-efficients are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$,for n=1,2,3,......

Examples: Expand f(x)=x in Fourier Series on the interval $-\pi\le x\le\pi$.

Sol. Observe that f(x)=x is bounded and integrable on $-\pi\le x\le\pi$,since it is continuous there. Further

$f'(x) = 1 > 0$ indicates that f(x) is monotone increasing on the entire interval. We extend this by defining f(x+2π)=f(x) for all values of x. This is a periodic function of period 2π. Its Fourier series corresponding to f(x) is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where the Fourier co-efficient are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\,dx = 0$

Since $x \cos nx$ and x are odd functions ,and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\,dx$$

Since $x \sin nx$ is even. Thus

$$b_n = \frac{2}{\pi}\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\frac{\cos nx}{n}dx$$

$$= -\frac{2}{n}\cos n\pi = \begin{cases} -\frac{2}{n}, & n \quad \text{even} \\ \frac{2}{n}, & n \quad \text{odd} \end{cases}$$

Hence f(x)=x generates Fourier Series in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots..$$

$$= 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right\}$$

Dirichlet's conditions:

A function f(x) will be said to satisfy Dirichlet's condition on an interval $-\pi \le x \le \pi$ in
Which it is defined when it is subjected to one of the two two following conditions:
(i)f(x) is bounded in [-T,T] and the interval [-T,T] can be decomposed in a finite number of sub-intervals such that f(x) is monotonic (increasing or decreasing) on each of the sub-intervals.
(ii) f(x) has a finite number of points of infinite discontinuity in [-T,T].
When arbitrary small neighbourhoodof these points are excluded from [-T,T] f(x) becomes bounded in the remaining part and this remaining part can be decomposed into a finite number of sub-intervals such that f(x) is monotonic in each of the sub-intervals. Moreover the improper integral $\int_{-\pi}^{\pi} f(x)dx$ is absolutely convergent.

1. **Convergence:**

When f(x) satisfies Dirichlet's condition on $-\pi \le x \le \pi$, the Fourier Series corresponding to f(x) converges to f(x) at any point x on $-\pi \le x \le \pi$ when f(x) is continuous and converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ when there is an ordinary discontinuityat the point. In particular at $x = \pi$ and $x = \pi$ it converges to $\frac{1}{2}\{f(-\pi+0)+f(\pi-0)\}$ when $f(-\pi+0)$ and $f(\pi-0)$ exist.

Example: Let f(x)= x-3 , $-3 \le x \le 0$

=3-x, $0 < x \le 3$.

f(x) is bounded in [-3,3]. The interval[-3,3] is decomposed as $[-3,0]\cup[0,3]$ such that f(x) is increasing in [-3,0] and decreasing in [0,3]. So we conclude this function f(x) satisfies Dirichlet's condition.

Lecture 4:

Fourier Series of a function with its periodic extension:

We introduce Fourier Series of a function f(x) which is primarily defined on the interval [-T,T] and then extending it to a periodic wave. But the function may appear as defined primarily on an interval [c,c+2T] where c may be any real number. In that case also we have no trouble of getting its Fourier Series.

Theorem: If f(x) be defined and integrable in [c,c+2T] and f(x+2T)=f(x) for all values of x , then the Fourier Series of f(x) is also

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T}\right)$$

Where the Fourier co-efficients are

Where c may be anyreal number.

Half Range Fourier Series:

A trigonometric series like the fourier Series is called a Half Range Fourier Series if only sine terms or only cosine terms are present. When only sine terms are present the series is called Half Range Sine Series; when only cosine terms are present the series is called Half Range Cosine Series.

When a half range series corresponding to a function is desired, the function is generally defined in the interval (0, T) which is half of the interval (-T, T).

Construction of Half range Sine Series:

Let f(x)be a function defined and integrable on the interval (0, T). We extend the domain of definition to [-T,0] defining by f(-x)=-f(x). This extension is shown in the adjacent figure. Then this extended f(x) becomes odd in the interval [-T, T].

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx$$

$$= 0 \qquad [\therefore f(x) \text{ is odd.}]$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx$$

$$= 0 \quad [\because f(x)\cos\frac{n\pi x}{T} \text{ is an odd function}]$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx$$

$$= \frac{2}{T}\int_{0}^{T} f(x)\sin\frac{n\pi x}{T}dx \quad [\because f(x)\sin\frac{n\pi x}{T} \text{ is an even function}].$$

The Fourier Series of f(x) becomes

$$\frac{0}{2} + \sum_{n=1}^{\infty}(0.\cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T})$$

$$\text{i.e.} \sum_{n=1}^{\infty}(b_n \sin\frac{n\pi x}{T})$$

which is the required Half Range Sine Series. Obviously if f(x) satisfies Dirichlets condition in [0, T] then this series is convergent and the value is as for Fourier Series.

Construction of Half range Cosine Series:

Let f(x)be a function defined and integrable on the interval (0, T). We extend the domain of definition to [-T,0] defining by f(-x)=f(x). This extension is shown in the adjacent figure. Then this extended f(x) becomes odd in the interval [-T, T].

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx = \frac{2}{T}\int_{0}^{T} f(x)dx \; [\because f(x) \text{ is even.}]$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx$$

$$= \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}dx \quad [\because f(x)\cos\frac{n\pi x}{T} \text{ is an even function}]$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx = 0 \quad [\because f(x)\sin\frac{n\pi x}{T} \text{ is an odd function}]$$

Consequently the Fourier Series of f(x) becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + 0.\sin\frac{n\pi x}{T})$$

i.e. $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{T}$.

Which is the required Half Range Cosine Series.

Here also this series converges according as f(x) satisfies Dirichlet's Condition.

Lecture 5:

In particular if the interval of definition becomes $[0,\pi]$ then

(1) The half Range Sine series becomes

$\sum_{n=1}^{\infty}(b_n \sin nx)$ **where** $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

(2) The half Range Cosine series becomes $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ **where**

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ **and** $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$**.**

Example: Consider the function f(x)=sinx , $0 < x \le \pi$. To get its Half Range Cosine series we extend the function to the interval ($-\pi$,0) defining by f(-x)=f(x). With this extension f(x) becomes even on the interval ($-\pi,\pi$). Then

$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin dx = \frac{4}{\pi}$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx \qquad (1)$$

$$= \frac{1}{\pi}\int_0^{\pi} \{\sin(x+nx) + \sin(x-nx)\}dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \text{for } n \neq 1$$

$$= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} = \frac{2(1+\cos n\pi)}{\pi(1-n^2)} \text{for } n \neq 1$$

Now $a_1 = 0$ (from (1)) and $b_n = 0$.

Therefore, the Half Range Cosine series becomes

$$\frac{1}{2}.\frac{4}{\pi}\sum_{n=2}^{\infty}\left(\frac{2(1+\cos n\pi)}{\pi(1-n^2)}\cos nx + 0.\sin nx\right)$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{2(1+\cos n\pi)}{\pi(1-n^2)}\cos nx\right)$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots\dots\dots\dots\right)$$

Parseval's Identity

If the Fourier Series of a function f(x) converges uniformly to f(x) in the interval (-T,T) then $\frac{1}{T}\int_{-T}^{T}\{f(x)\}^2 dx = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a^2{}_n + b^2{}_n)$

Where a_n, b_n are Fourier Co-efficients of f(x).

Note: 1) Corresponding to Half Range sine series $\sum_{n=1}^{\infty} b_n \sin nx$ the Persaval's indentity would be $\frac{2}{T}\int_0^T\{f(x)\}^2 dx = \sum_{n=1}^{\infty} b^2{}_n$ since here f(x) is extended to an even function.

2) Corresponding to Half Range cosine series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$ the Persaval's indentity would be $\frac{2}{T}\int_0^T \{f(x)\}^2 dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty} a_n^2$ since here f(x) is extended to an odd function i.e. $\{f(x)\}^2$ is extended to even.

Example: Consider the function f(x)= -x, $-2 < x \le 0$

$= x,\ 0 \le x \le 2.$

Find the Fourier series of this function.

Solution: We see f(x) is an even function. Extending this to a periodic function defined by f(x+4)=f(x).

Here $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} -xdx + \frac{1}{2}\int_{0}^{2} xdx = 2$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{2}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx$ [since $f(x)\cos\frac{n\pi x}{2}$ is an even function].

$= \left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - 1\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$

$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$ for $n \ne 0$.

$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = 0$ [since $f(x)\sin\frac{n\pi x}{2}$ is an odd function]

Again the function f(x) satisfies Dirichet's condition and it is continuous everywhere

So its Parseval's Identity is

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2 dx = \frac{2^2}{2} + = \sum_{n=1}^{\infty}\left[\left\{\frac{4}{n^2\pi^2}.(\cos n\pi - 1)\right\}^2 + 0^2\right]$$

$$\text{Or, } \frac{1}{2}\left[\int_{-2}^{0}(-x)^2 dx + \int_{0}^{2} x^2 dx\right] = 2 + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}.(\cos n\pi - 1)^2$$

$$\text{Or, } \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2} = 2 + \frac{64}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\dots\dots\right)$$

$$\text{Or, } \frac{8}{3} = 2 + \frac{64}{\pi^2}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\dots\dots\right)$$

Or, $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+..............=\frac{\pi^4}{96}$

Lecture 6

Introduction to Fourier Transform

In the previous chapter we have seen if a function f(x) satisfies dirichlel's condition on the interval (-T, T) then it can be expanding in the Fourier Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T})$$

Where a_n, b_n are Fourier coefficients. Using the Euler's identity $e^{i\theta}=\cos\theta+i\sin\theta$ the Fourier Series of f(x) can be written as

$$f(x)=\sum_{n=1}^{\infty}c_n e^{\frac{in\pi x}{T}} \qquad (1)$$

Where $c_n=\frac{1}{2T}\int_{-T}^{T}f(x)e^{-\frac{in\pi x}{T}}dx$ are also known as fourier coefficients. If now this hold for all values of T as $T\to\infty$, the expansion (1) takes the form f(x)=$=\int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}F(s)\right\}e^{-isx}ds$

Where F(s)=$=\int_{-\infty}^{\infty}f(t)e^{ist}dt$ which is known as Fourier Transform.

Fourier Transforms of a function:

It transforms an integrable function to an another function defined as follows

Let f(x) be integrable function on any interval (-T, T). Then the improper integral

$$\Im(f(x))=F(s)=\int_{-\infty}^{\infty}f(x)e^{isx}dx$$

is called the Fourier Transform of the function f. This is a function of s.

Fourier Sine and Cosine Integral Theorem:

The Fourier sine transform of f(x) , $0<x<\infty$ is defined as

$$\Im(f(x)) = F_s(s) = \int_{-\infty}^{\infty} f(x)\sin(sx)dx$$

The Fourier Cosine transform of f(x) , $0 < x < \infty$ is defined as

$$\Im(f(x)) = F_c(s) = \int_{-\infty}^{\infty} f(x)\cos(sx)dx$$

provided the improper integral are convergent.

Illustration:

(i) Let us consider the function

f(x)=1 , $-1 \le x \le 1$

=0 , otherwise

It's Fourier Transform,

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \int_{-\infty}^{-1} 0.e^{isx}dx + \int_{-1}^{1} 1.e^{isx}dx + \int_{1}^{\infty} 0.e^{isx}dx = \int_{-1}^{1} 1.e^{isx}dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1} = \frac{e^{is} - e^{-is}}{is} = \frac{(\cos s + i\sin s) - (\cos s - i\sin s)}{is}$$

$$= \frac{2\sin s}{s} \text{ for } s \ne 0.$$

For, s=0, $F(0) = \int_{-1}^{1} 1.e^{0.x}dx = \int_{-1}^{1} dx = 2$

So, the Fourier Transform of f(x) is

$$F(s) = \frac{2\sin s}{s}, \qquad s \ne 0$$

$$=2, \qquad s=0$$

<u>Fourier Cosine & Sine Transforms</u>:

1. If f(x), $-\infty < x < \infty$ is an even function then its Fourier Transform $F(s) = 2F_c(s)$ where $F_c(s)$ is the Fourier cosine transform of f(x).

Proof: $F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \int_{-\infty}^{\infty} f(x)(\cos sx + i\sin sx)dx$$

$$= \int_{-\infty}^{\infty} f(x)\cos sx dx + i\int_{-\infty}^{\infty} f(x)\sin sx dx$$

$$= 2\int_0^\infty f(x)\cos sx\,dx + i.0$$

[since $f(x)\cos sx$ is an odd function and f(x)sinsx is odd]

2. If f(x), $-\infty < x < \infty$ is an odd function then its Fourier Transform $F(s) = 2iF_s(s)$ where $F_s(s)$ is the Fourier sine transform of f(x).

Illustration:

Let us consider the function f(x)=1 , -1<x<1

=0 , otherwise.

is an even function. So its Fourier Transform, $F(s) = 2F_c(s)$

$$\text{or, } F_c(s) = \frac{1}{2}F(s)$$

So, its Fourier Cosine Transform,

$$F_c(s) = \frac{1}{2}.\frac{2\sin s}{s}, \quad s \neq 0$$

$$= \frac{1}{2}.2 , \qquad s=0$$

$$\text{i.e. } F_c(s) = \frac{\sin s}{s}, \quad s \neq 0$$

=1, s=0.

Lecture 7:

Fourier Cosine & Sine Transforms of elementary functions:

Transform of some standard function:

The Fourier transform are

(1) $F(s) = \sqrt{2\pi}e^{-\frac{s^2}{2}}$ of $f(x) = e^{-\frac{x^2}{2}}$

(2) $F(s) = \dfrac{2a}{a^2+s^2}$ of $f(x) = e^{-a|x|}$

(3) $F(s) = \dfrac{2\sin sa}{s}$, $s \neq 0$ of f(x)=1, $|x| < a$

=2a , s=0 =0 , $|x| > a$

(4) $F_s(s) = \dfrac{s}{a^2+s^2}$ of $f(x) = e^{-ax}\,(a > 0)$

(5) $F_c(s) = \dfrac{a}{a^2 + s^2}$ of $f(x) = e^{-ax}\ (a > 0)$

Proofs.

(1) The Fourier Transform corresponding to f(x) is

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx} dx = \int_{-\infty}^{\infty} e^{\frac{-x^2+2isx}{2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2-2isx}{2}} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2-2isx+(is)^2+s^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2+s^2}{2}} dx = e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} dx$$

[putting t=x-i.s i.e. dt=dx]

$$= e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = 2e^{-\frac{s^2}{2}} \int_{0}^{\infty} e^{-\frac{t^2}{2}} dt$$

$$= 2\sqrt{2} e^{-\frac{s^2}{2}} \int_{0}^{\infty} e^{-z^2} dz \text{ [putting } \frac{t}{\sqrt{2}} = z, \text{ i.e. } \frac{1}{\sqrt{2}} dt = dz,]$$

$$= 2\sqrt{2} e^{-\frac{s^2}{2}} \frac{\sqrt{\pi}}{2} = \sqrt{2\pi} e^{-\frac{s^2}{2}}$$

Properties of Fourier Transform:

(1) **Linear Property**: If f(x) and g(x) are two functions having Fourier Transforms then

$\mathfrak{I}(c_1 f + c_2 g) = c_1 \mathfrak{I}(f) + c_2 \mathfrak{I}(g)$

where c_1 and c_2 are constants.

Proof: $\mathfrak{I}(c_1 f + c_2 g) = \int_{-\infty}^{\infty} \{c_1 f(x) + c_2 g(x)\} e^{isx} dx$

$= c_1 \int_{-\infty}^{\infty} f(x) e^{isx} dx + c_2 \int_{-\infty}^{\infty} g(x) e^{isx} dx$

$= c_1 \mathfrak{I}(f) + c_2 \mathfrak{I}(g).$

(2) **Change of Scale Property:** If the Fourier Transforms of f(x) , $\mathfrak{I}\{f(x)\} = F(s)$ then the Fourier Transforms of f(ax) $\mathfrak{I}\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof: Since $F\{f(x)\} = F(s)$ therefore F(s)= $\int_{-\infty}^{\infty} f(x) e^{isx} dx$ …….(1)

Now $\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{isx}dx$ [Putting t=ax i.e. dt=adx]

$$=\frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{\frac{ist}{a}}dt$$

$$=\frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{\frac{isx}{a}}dx=\frac{1}{a}F\left(\frac{s}{a}\right) \quad \text{by (1)}$$

(3) **<u>Shifting Property</u>:** If the Fourier Transforms of f(x), $\mathcal{F}\{f(x)\} = F(s)$ then the Fourier Transforms of f(x-a),

$$\mathcal{F}\{f(x-a)\} = e^{ias}F(s)$$

Proof: $\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{isx}dx$

$$=\int_{-\infty}^{\infty} f(t)e^{is(a+t)}dt \quad \text{[put x-a=t } \therefore dx = dt]$$

$$=e^{isa}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$=e^{isa}\int_{-\infty}^{\infty} f(x)e^{isx}dx=e^{ias}F(s).$$

(4) **<u>Modulation Property</u>:** If the Fourier Transforms of f(x) , $\mathcal{F}\{f(x)\} = F(s)$ then the Fourier Transforms of

$$\mathcal{F}\{f(x)\cos ax\} = \frac{1}{2}\{F(s-a) + F(s+a)\},$$

Proof: $\mathcal{F}\{f(x)\cos ax\} = \int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}dx$

$$=\int_{-\infty}^{\infty} f(x)e^{isx}\frac{e^{iax}+e^{-iax}}{2}dx$$

$$=\frac{1}{2}\left\{\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx + \int_{-\infty}^{\infty} f(x)e^{i(x-a)x}dx\right\}$$

$$=\frac{1}{2}\{F(s+a) + F(s-a)\}=\frac{1}{2}\{F(s-a) + F(s+a)\}$$

Illustration: Let we are to evaluate the Fourier Transforms of the function

$$f(x)=4e^{-|x|} - 5e^{-3|x+2|}$$

$$\mathcal{F}\{4e^{-|x|} - 5e^{-3|x+2|}\}=4\mathcal{F}(e^{-|x|})-5\mathcal{F}(e^{-3|x+2|}) \quad \text{.........(1)}$$

Now, by formula , $\mathcal{F}(e^{-|x|})=\frac{2}{s^2+1}$

By shifting property, $\mathcal{F}(e^{-3|x+2|}) = \mathcal{F}(e^{-3|x-(-2)|})$

$$=e^{-2is}\mathcal{F}(e^{-3|x|})$$

$$=e^{-2is}\mathcal{F}(e^{-|3x|})$$

$$=e^{-2is}\frac{1}{3}F\left(\frac{s}{3}\right) \text{, by change of scale property}$$

Where F(s)= $\mathcal{F}(e^{-|x|}) = \frac{2}{s^2+1}$

$$\therefore F\left(\frac{s}{3}\right) = \frac{2}{\left(\frac{s}{3}\right)^2+1}=\frac{18}{s^2+9} = \frac{2}{s^2+1}$$

$\therefore \beth\,(e^{-3|x+2|}) = \frac{e^{-2is}}{3} \cdot \frac{18}{s^2+9} = \frac{6e^{-2is}}{s^2+9}$

So , from (1) we get , $\beth\left\{4e^{-|x|} - 5e^{-3|x+2|}\right\} = 4.\frac{2}{s^2+1} - 5.\frac{6e^{-2is}}{s^2+9}$

$= \frac{8}{s^2+1} - \frac{30e^{-2is}}{s^2+9}$

Lecture 8

(5) **Fourier Transform of Derivatives:** If f(x), $f'(x), f''(x), \dots\dots\dots f^{(n-1)}(x)$ all tend to 0 as $x \to \pm\infty$ and $\int_{-\infty}^{\infty}\left|f^{(j)}(x)\right|$dx converges for all j then $\beth\left\{f^{(n)}(x)\right\} = (-is)^n \beth\{f(x)\}$

On multiplication by x.

If $\beth\{f(x)\} = F(s)$ then $\beth\{xf(x)\}$ =-i$F'(s)$

Proof: $\beth\{xf(x)\} = \int_{-\infty}^{\infty} xf(x)e^{isx}\,dx$

Now $F'(s) = \frac{d}{ds}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$

$= \int_{-\infty}^{\infty} ixf(x)e^{isx}\,dx$

$= i\int_{-\infty}^{\infty} xf(x)e^{isx}\,dx$

$= i\beth\{xf(x)\}$

$\therefore \beth\{xf(x)\} = -i\frac{d}{ds}\beth\{f(x)\}$

Example: Find the Fourier transform of $\beth\left\{xe^{-4x^2}\right\}$

Solution: Now $\beth\left\{e^{-4x^2}\right\} = \beth\left\{e^{-\frac{(\sqrt{8}x)^2}{2}}\right\}$

Therefore , $\beth\left\{e^{-\frac{x^2}{2}}\right\} = \sqrt{2\pi}e^{-\frac{s^2}{2}}$

$\therefore \beth\left\{e^{-\frac{(\sqrt{8}x)^2}{2}}\right\} = \frac{1}{\sqrt{8}}\sqrt{2\pi}e^{-\frac{\left(\frac{s}{\sqrt{8}}\right)^2}{2}}$ (by change of scale property)

$= \frac{1}{\sqrt{8}}\sqrt{\pi}e^{-\frac{s^2}{16}}$

$\therefore \beth\left\{e^{-4x^2}\right\} = \frac{\sqrt{\pi}}{2}e^{-\frac{s^2}{16}}$ …………..(1)

Or, $\mathfrak{F}\left(\frac{d}{ds}\left(e^{-4x^2}\right)\right) = (-is)\mathfrak{F}\{e^{-4x^2}\}$

Or, $\mathfrak{F}(-8xe^{-4x^2})=-is\mathfrak{F}\{e^{-4x^2}\}$

Or, $-8\,\mathfrak{F}(xe^{-4x^2})=-is.\frac{\sqrt{\pi}}{2}e^{-\frac{s^2}{16}}$ by (1)

Or, $\mathfrak{F}\{xe^{-4x^2}\} = \frac{is\sqrt{\pi}}{2}e^{-\frac{s^2}{16}}$

(6) <u>Convolution Theorem</u>: If f(x) and g(x) are two functions defined on the interval $(-\infty, \infty)$ then $f*g=\int_{-\infty}^{\infty} f(u)g(x-u)du$ is called the convolution of the two functions f and g.

The Fourier Transform of the convolution of f(x) and g(x) is the product of their Fourier Transform

i.e. $\mathfrak{F}(f*g)= \mathfrak{F}\{f(x)\}\mathfrak{F}\{g(x)\}$

Example: Find the function f(x) from the following integral equation

$$f(x)=g(x)+\int_{-\infty}^{\infty} f(u)h(x-u)du$$

Solution: Taking Fourier Transform on both side we get

$$\mathfrak{F}\{f(x)\} = \mathfrak{F}\{g(x)\} + \mathfrak{F}\{\int_{-\infty}^{\infty} f(u)h(x-u)du\} = \mathfrak{F}\{g(x)\} + \mathfrak{F}(f*h)$$

$$= G(s) + \mathfrak{F}\{f(x)\}\mathfrak{F}\{h(x)\} \text{ , by Convolution theorem}$$

Or, F(s)=G(s)+F(s)H(s)

Or, $F(s)=\frac{G(s)}{1-H(s)}$ $\quad\therefore \mathfrak{F}\{f(x)\} = \frac{G(s)}{1-H(s)}$

So by Fourier integral theorem

$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(s)}{1-H(s)}\, e^{-isx}ds = f(x)$ where f(x) is continuous.

$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(s)}{1-H(s)}\, e^{-isx}ds$ at the point of continuity.

<u>Lecture 9</u>

<u>Inverse Fourier transform or Fourier Integral</u>:

In this article we discuss how a function can be found if its Fourier Transform is known. This is due to the following theorem.

Theorem 1: (Fourier integral theorem)

If a function f(x)

(i)satisfies Dirichlet's condition in every finite interval [-T,T]

(ii) $\int_{-\infty}^{\infty}|f(x)|dx$ exists finitely

Then $\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)\, e^{-isx}ds = f(x)$ where f(x) is continuous

$=\frac{1}{2}\{\lim_{t\to x+}f(t) + \lim_{t\to x-}f(t)\}$ when f(x) is discontinuous at x

Where F(s) is the Fourier Transform of f(x).

Definition: The integral $\mathfrak{I}^{-1}(F(s)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx} ds)$ is called Inverse Fourier Transform or Fourier Integral of the function f(x) , where F(s) is the Fourier Transform of f(x)

So, $\mathfrak{I}^{-1}(F(s)) = f(x)$ at the point of continuity.

Fourier Sine and Cosine Integral Theorem:

Theorem: If a function (i) satisfies Dirichlet's condition in every finite interval [0, T]

(ii) (ii) $\int_{-\infty}^{\infty} |f(x)| dx$ exists finitely

Then

(a) $\frac{2}{\pi}\int_0^{\infty} F_s(s) sinsxds = f(x)$ where f(x) is continuous

$=\frac{1}{2}\{\lim_{t\to x+} f(t) + \lim_{t\to x-} f(t)\}$ where f(x) is discontinuous

Where $F_s(s)$ is the Fourier Sine Transform of f(x).

(b) $\frac{2}{\pi}\int_0^{\infty} F_c(s) cossxds = f(x)$ where f(x) is continuous.

$=\frac{1}{2}\{\lim_{t\to x+} f(t) + \lim_{t\to x-} f(t)\}$ where f(x) is discontinuous

Where $F_c(s)$ is the Fourier Cosine Transform of f(x).

Fourier Sine Integral: The integral $\mathfrak{I}^{-1}{}_s(F_s(s)) = \frac{2}{\pi}\int_0^{\infty} F_s(s) sinsxds$ is called Inverse Fourier Transform or Fourier Sine Inverse of f(x) , where $F_s(s)$ is Fourier Sine Transform of f(x).

Fourier Cosine Integral: integral $\mathfrak{I}^{-1}{}_c(F_c(s)) = \frac{2}{\pi}\int_0^{\infty} F_c(s) cossxds$ is called Inverse Fourier Transform or Fourier Cosine Inverse of f(x) , where $F_c(s)$ is Fourier Cosine Transform of f(x).

Inverse Property of Inverse Fourier Transform: If F(s) and G(s) are Fourier transform of th function f(x) and g(x) respectively then

$$\mathfrak{I}^{-1}\{c_1 F(s) + c_2 G(s)\} = c_1 \mathfrak{I}^{-1}\{F(s)\} + c_2 \mathfrak{I}^{-1}\{G(s)\}$$

Where c_1, c_2 are constants.

Examples:

1. Find the Fourier transform of the function

f(x)=1-x^2 , $|x| \leq 1$

=0 , $|x| > 1$

Hence evaluate $\int_{-\infty}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2} dx$

The Fourier Transform,

F(s)= $\int_{-\infty}^{\infty} f(s)\, e^{isx} ds = \int_{-1}^{1} (1-x^2)\, e^{isx} dx$

$=\int_{-1}^{1} (1-x^2)(\cos sx + i\sin sx) dx$

$=\int_{-1}^{1} (1-x^2)\cos sx dx + i\int_{-1}^{1} (1-x^2)\sin sx dx$

$=2\int_{-1}^{1} (1-x^2)\cos sx dx$

[$\therefore (1-x^2)\cos sx$ *is even and* $(1-x^2)\sin sx$ *is odd function*

$= 2\{[(1-x^2)\frac{\sin sx}{s}]_0^1 + 2\int_0^1 2x\frac{\sin sx}{s} dx\}$

$=\frac{4}{s}\left\{-\frac{\cos s}{s} + \frac{1}{s}\left[\frac{\sin sx}{s}\right]_0^1\right\}$

$=\frac{4}{s}\left\{-\frac{\cos s}{s} + \frac{\sin s}{s^2}\right\} = \frac{4\{s\cos s - \sin s\}}{s^3}$

Since f(s) is an even function therefore F(s)=2$F_c(s)$

$$F_c(s) = \frac{1}{2}F(s) = -\frac{2\{s\cos s - \sin s\}}{s^3}$$

By Fourier integral theorem,

$\frac{2}{\pi}\int_0^{\infty} F_c(s)\cos sx ds = f(x)$

Or, $\frac{2}{\pi}\int_0^{\infty} -\frac{2\{s\cos s - \sin s\}}{s^3}\cos sx ds = f(x)$

Or, $\frac{4}{s}\int_0^{\infty} -\frac{\{s\cos s - \sin s\}}{s^3}\cos sx ds = f(x)$=1-$x^2$, $|x| \leq 1$

=0 , $|x| > 1$

Putting x=$\frac{1}{2}$ we get

$\frac{-4}{\pi}\int_0^{\infty} \frac{\{s\cos s - \sin s\}}{s^3}\cos\frac{s}{2} ds$=1-$\frac{1}{4}$=$\frac{3}{4}$

Or, $\int_0^{\infty} \frac{\{scoss - sins\}}{s^3} \cos\frac{s}{2} ds = -\frac{3\pi}{16}$

i.e. , , $\int_0^{\infty} \frac{\{scosx - sinx\}}{x^3} \cos\frac{x}{2} ds = -\frac{3\pi}{16}$.

Lecture 10:

2. Find the Fourier Cosine transform of the function

f(x)=cosx , 0<x<a

=0 ,x>a

Solution: The required transform,

$$F_c(s) = \int_0^{\infty} f(x) cossxdx$$

$$= \int_0^{a} cosxcossxdx$$

$$= \frac{1}{2} \int_0^{a} \cos(1+s)x + \cos(1-s)xdx$$

$$= \frac{1}{2}\left\{\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}\right\}$$

3. Find the Fourier Sine transform of the function f(x)=$\frac{1}{x}$

Solution: The required transform,

$$F_s(s) = \int_0^{\infty} f(x) sinsxdx = \int_0^{\infty} \frac{sinsx}{s} dx$$

$$= \frac{1}{s} \int_0^{\infty} \frac{ssint}{t} dt$$ putting sx=t i.e. , dx=$\frac{1}{s} dt$

$$= \int_0^{\infty} \frac{sint}{t} dt = \frac{\pi}{2}$$

4. Find the function whose cosine transform is $\frac{sinas}{s}$

Solution: Let f(x) be the required function. Then $F_c(s) = \frac{sinas}{s}$ corresponding to f(x).

By Fourier cosine integral theorem,

$\frac{2}{\pi} \int_0^{\infty} F_c(s) cossxds = f(x)$ where f(x) is continuous

$$or, \frac{2}{\pi} \int_0^{\infty} \frac{sinas}{s} cossxds = f(x)$$

Or, $f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{2sinascossx}{s} ds = \frac{1}{\pi} \int_0^{\infty} \frac{\sin(a+x)s + \cos(a-x)s}{s} ds$

$$=\frac{1}{\pi}\left\{\int_0^{\infty}\frac{\sin(a+x)s}{s}ds+\int_0^{\infty}\frac{\sin(a-x)s}{s}ds\right\}$$

Or, f(x)= $\frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$ if x<a

$=\frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right)$ if x>a since, $\int_0^{\infty}\frac{\sin sx}{x}dx=\frac{\pi}{2}$

EXERCISES

1. Find the Fourier Transform of

   $f(x)=x^2$, $|x|<a$

   $=0$, $|x|\geq a$

2. If the Fourier transform, $F(s)=e^{-2|x|}$ evaluate f(x)
3. Find Fourier Cosine transform of $f(x)=\frac{1}{1+x^2}$ and hence derive fourier sine transform of the function $\frac{x}{1+x^2}$
4. Evaluate $ב\left\{\frac{\cos 3x}{x^2+2}\right\}$
5. Evaluate the integral $\int_0^{\infty}\frac{s\sin sx}{1+s^2}ds$
6. Prove that $\int_0^{\infty}\frac{s^3\sin sx}{4+s^4}ds=\frac{\pi}{2}\sin x$,$0\leq x\leq\pi$

   $=0$, $x>\pi$

7. Find Fourier sine and Cosine transform of e^{-x} and using the inversion formula recover the original function, in both the cases.
8. Verify the Convolution theorem for the function

   $f(x)=1$, $|x|<1$

   $=0$, $|x|>1$

   And $g(x)=1$, $|x|<1$

   $=0$, $|x|>1$

9. Find the Fourier Integral of the function

   $f(t)=1$ $|t|\leq 1$

   $=0$, $|t|>1$

10. Evaluate $ב^{-1}\left\{e^{-|s|}\cos s\right\}$

MODULE-II

(Probability Distributions)

Lecture 11:

Introduction: In Statistics, something is random when it varies by chance. For example, when rolling a six sided die there are six equally possible outcomes, the observed outcome on any one roll is random. The variation of a random event such as rolling a die can be described by the probability distributions that we will see in this lesson.

Random variable: a numerical characteristic that takes on different values due to chance

Examples:

Throwing Coins.
The number of heads in four flips of a coin (a numerical property of each different sequence of flips) is a **random variable** because the results will vary between trials.

Heights of population.
Sample of 100 are repeatedly pulled from the population of all Penn State students and their heights are measured. The mean height of samples of 100 Penn State students is a **random variable** because the statistic will vary between samples. While most sample means will be similar to the population mean, they will not all equal the population mean due to random sampling variation.

Random variables are classified into two broad types: discrete and continuous, **discrete random variable and continuous random variable.**

Discrete Random Variable : A random variable X is said to be discrete if the spectrum of X is finite or countably infinite i.e. an infinite sequence of distinct values.

Continuous Random Variable: A random variable X is said to be continuous if it can assume every value in an interval.

Illustrations.

(i) Let us consider the random experiment of tossing two (unbiased) coins. Then the sample space S contains 4 sample points.

i.e. , S= { HH , HT, TH, TT}

Let the random variable X be such that X (an outcome) = "the number of heads". Then X is a function over S defined by

X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 0.

Thus the spectrum of X is { 0, 1, 2} which is a finite set. Hence X is a discrete random variable here. Here the event (X=1) = { TH, HT} = 'One Head' , the event ($-1 < X \leq 0$) = {TT} = 'Two tails'.

(ii).Let the random variable X denote the weights (in kg) of a group of individuals. Then X can assume every value in an interval say (30, 100), supposing there is no individual having weight less than 30 and greater than 100. Hence X is a continuous random variable. Here event ($42 < X \leq 50$) = the group of individual whose weight lie between 42 and 50, including 50; the event (X = 70) = The group of individuals whose weight is 70 kg.

Probability Mass Function and Discrete Distribution:

Let X be a discrete random variable which assumes the values $x_0, x_1, x_2, \ldots, x_n, \ldots$ Let $P(X = x_i) = f(x_i) = f_i$. So the value of f_i depends on x_i i.e. i. Thus the function f_i is called *Probability Mass Function* **(p.m.f)** of the random variable X. a particular value of f_i is called probability mass.

The set of ordered pairs (x_i, f_i) is called discrete probability distribution of the random variable X.

Discrete distribution is presented in the following way :

X:	0	1	2	…	…
f_i:	f_0	f_1	f_2	…	…

Illustration. For the random experiment of tossing two coins given in illustration (i), we see X assumes the values 0, 1 and 2.

Moreover $(X = 0) = \frac{1}{4}, P(X = 1) = \frac{1}{2}, P(X = 2) = \frac{1}{4}$.

So, the distribution of the number of heads is given by

X:	0	1	2	…	…
f_i:	1/4	*1/2*	1/4	…	…

Fundamental Properties of pmf:

If

X:	0	1	2	…	…
f_i:	f_0	f_1	f_2	…	…

is a discrete distribution of X, then the pmf has the following two properties:

(i) $f_i \geq 0$

(ii) $\sum_i f_i = 1$

Lecture 12:

Distribution Function or Cumulative Distribution Function:

The distribution function (d.f) of a random variable X (discrete or continuous) is given by

$$F(x) = P(-\infty < X < x), -\infty < x < \infty$$

Thus, if $x_i \leq x \leq x_{i+1}$, then

$$F(x) = P(X = x_0) + P(X = x_i) + \cdots + P(X = x_i) = \sum_{\alpha=0}^{i} f_\alpha \text{ .}$$

Illustration. In the discrete distribution

X:	0	1	2
f_i:	1/4	*1/2*	1/4

The distribution function is

$$F(x) = 0, \quad x < 0$$

$$= 1/4, \quad 0 \le x < 1$$

$$= (1/4) + (1/2), \quad 0 \le x < 2$$

$$= (1/4) + (1/2) + (1/4), \quad 2 \le x$$

Properties of Distribution Function:

(i) The distribution function F(x) is a monotonic non-decreasing function.

(ii) $F(-\infty) = 0$ and $F(\infty) = 1$ and $0 \le F(x) \le 1$.

(iii) $F(x)$ is a continuous on the right of all points and has a jump discontinuity on the left at x=a , the height of jump being equal to $P(X = a) i.e., \lim_{x \to a^+} F(a) = F(a)$ and $F(a) = \lim_{x \to a^-} F(x) = P(x = a)$

(iv) Suppose a and b are any real numbers such that $a<b$
Then $P(a < X \le b) = F(b) - F(a)$,
$P(a < X < b) = F(b) - F(a) - P(X=b)$
And $P(a \le X < b) = F(b) - F(a) - P(X=b) + P(X=a)$

Illustration. (i) Let X be a random variable denoting the number of points appearing in a throwing of a die. The distribution of X is

X:	1	2	3	4	5	6
f_i:	1/6	1/6	1/6	1/6	1/6	1/6

Now, if $x<1$, $F(x) = P(X \le x) = 0$

If $1 \le x < 2$, $F(x) = P(X \le x) = f_1 = \frac{1}{6}$

If $2 \le x < 3$, $F(x) = P(X \le x) = f_1 + f_2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

and so on.

Thus the distribution function *F(x)* is given by :

$F(x) = 0 , \quad -\infty < x < 1$

$= 1/6 , \quad 1 \le x < 2$

$= 2/6 , \quad 2 \le x < 3$

$= 3/6 , \quad 3 \le x < 4$

$= 4/6 , \quad 4 \le x < 5$

$= 5/6 , \quad 5 \le x < 6$

$= 1 , \quad 6 \le x < \infty$

Probability Density Function:

For continuous random variables, as we shall soon see, the probability that X takes on any particular value x is 0. That is, finding $P(X = x)$ for a continuous random variable X is not going to work. Instead, we'll need to find the probability that X falls in some interval (a, b), that is, we'll need to find $P(a < X < b)$. We'll do that using a probability density function ("p.d.f."). We'll first motivate a p.d.f. with an example, and then we'll formally define it.

Definition. The **probability density function ("p.d.f.")** of a continuous random variable X with support S is an integrable function $f(x)$ satisfying the following:

(1) $f(x)$ is positive everywhere in the support S, that is, $f(x) > 0$, for all x in S

(2) The area under the curve $f(x)$ in the support S is 1, that is:

$$\int_S f(x)dx = 1$$

(3) If $f(x)$ is the p.d.f. of x, then the probability that x belongs to A, where A is some interval, is given by the integral of $f(x)$ over that interval, that is:

$$P(X \in A) = \int_A f(x)dx$$

Density Curve : The curve given by *y=f(x), (f(x) is pdf)* is the probability density curve which gives the graphical representation of the corresponding continuous distribution.

Illustration : consider a function *y=f(x) (f(x) is pdf)* which is defined as

$f(x) = \frac{2}{x^3} \; , 1 \le x < \infty$

= 0, elsewhere

As $f(x) > 0$ everywhere and $\int_{-\infty}^{\infty} f(x)dx$

$$= \int_{1}^{\infty} \frac{2}{x^3}dx = \lim_{P\to\infty} \int_{1}^{P} \frac{2}{x^3}\, dx = \lim_{P\to\infty} \left(1 - \frac{1}{P^2}\right) = 1 - 0 = 1$$

So this *f(x)* is a probability density function of some random variable.

Now, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0 \cdot dx = 0 \; when - \infty < x < 1$

And $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{1}^{x} \frac{2}{x^3} dx = 0 \; when \; 1 < x < \infty$

$= 1 - \frac{1}{x^2}$

So the distribution function of *pdf* is

$F(x) = 0, \quad -\infty < x < 1$

$= 1 - \frac{1}{x^2}\,, \quad 1 \le x < \infty$

Lecture 13:

Expectation of a Discrete Random Variable:

Let X be a discrete random variable whose distribution is

X:	0	1	2	… …	N	…
f_i:	f_0	f_1	f_2	… …	f_n	…

Then the mean or expectation or expected value of X , denoted by *E(X) or m(X)* or simply *m* is defined as

$E(X) = x_0 f_0 + x_1 f_1 + x_2 f_2 + \cdots = \sum_i x_i f_i$, provided the series is absolutely convergent if the above sum is an infinite series.

Expectation of Continuous Random Variable :

For a continuous random variable X with probability density function *f(x)* , the mean or expectation of X is defined as

$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$,

Provided the infinite integral converges absolutely.

Similarly, the mean of a function $\Psi(X)$ on the random variable X denoted by $E\{\Psi(X)\}$ is defined as

$E\{\Psi(X)\} = \sum_i \Psi(X_i) f_i$,   for a discrete distribution.

$= \int_{-\infty}^{\infty} \Psi(x) f(x) dx$,   for a continuous distribution

Illustration. (i) Suppose a die is rolled. Let X be the number of points on the die. Then its values are 1, 2, 3, 4, 5, 6.

So, P(X=i) = 1/6 for i = 1 ,2 , 3 ,4 ,5 ,6.

So the distribution X is

X:	1	2	3	4	5	6
f_i:	1/6	1/6	1/6	1/6	1/6	1/6

Therefore its expectation

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

And

$$E(X^{2+}1) = (1^2+1)\cdot\tfrac{1}{6} + (2^2+1)\cdot\tfrac{1}{6} + (3^2+1)\cdot\tfrac{1}{6} + (4^2+1)\cdot\tfrac{1}{6} + (5^2+1)\cdot\tfrac{1}{6} + (6^2+1)\cdot\tfrac{1}{6} = \tfrac{97}{6}$$

(iii) Let the *pdf* of a continuous random variable X is

$$f(x) = \frac{1}{2} \quad in - 1 < x < 1$$

$= 0, \quad$ elsewhere

The mean or expectation of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} x \frac{1}{2} dx = 0$$

And also, $E(2X^3) = \int_{-\infty}^{\infty} 2x^3 f(x) dx = \int_{-1}^{1} 2x^3 \cdot \frac{1}{2} dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$

Properties of Expectation

(i) $E(a) = a$, where a is being a constant
(ii) $E(aX) = a\,E(X)$ where a being a constant
(iii) $E(X \pm Y) = E(X) \pm E(Y)$, X, Y are two r.v.
(iv) $E(XY) = E(X)\,E(Y)$ if the two r.v X and Y are independent.

Variance and S.D :

The variance of a r.v. X , denoted by $Var(X)$ is defined as $Var\,(X) = E((X - m^2))$, where $m = E(X)$

The positive squre root of $Var\,(X)$ is called the *standard deviation*of X and is denoted by $\sigma(X)$ or σ_x or simply . Thus $\sigma = +\sqrt{var(X)}$.

Remarks:

(i) The variance describes how widely the probability masses are spread about the mean i.e. it gives an inverse measure of concentration of the probability masses about the mean which is called the measure of dispersion.
(ii) As $Var(X) = 0$ only when $X - m = 0$ i.e $= m$, so in that case whole mass is concentrated at the mean.

Theorem:

(i) $Var(X) = E(X^2) - m^2 = E(X^2) - \{E(X)\}^2$

(ii) $Var(aX + b) = a^2 Var(X)$

(iii) $Var(k) = 0$ where k is constant.

(iv) $Var(X) = E\{X(X-1)\} - m(m-1)$ where m is mean of X

Illustration: Consider the following distribution of a random variable X:

$f(x) = \frac{1}{2}x, \quad 0 \le x \le 2$

$= 0$, elsewhere

The expectation of X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{2} x\frac{1}{2}xdx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}$$

Now,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{2} x^2\frac{1}{2}xdx = \left[\frac{x^4}{8}\right]_0^2 = 2$$

Therefore , $Var(X) = E(X^2) - \{E(X)\}^2 = 2 - \frac{16}{9} = \frac{2}{9}$

So, $s.d = \sqrt{2}/3$

Ex. 1. Find the probability distribution of the number of heads when a fair coin is tossed repeatedly until the first tail appears.

The sample space corresponding to the random experiment of the tossing of the fair coin is *S= { T, HT, HHT, HHHT, ...}*

Let the random variable X denotes "the number of heads in the experiment until the first tail appears".

Then the spectrum of X is {0, 1, 2, 3, …}

Now $P(X = 0) = P(T) = \frac{1}{2}$

$P(X = 1) = P(HT) = P(H)P(T)$ [since trails are independent]

$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}$

$P(X = 2) = P(HHT) = P(H)P(H)P(T) = \frac{1}{2^3}$ and so on

Hence the probability distribution of X is

X :	0	1	2	3	$\cdots$
f_i :	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\cdots$

Ex. 2. A random variable X has the following probability mass function

X :	0	1	2	3	4	5	6	7
$P(X = k) = f(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i)determine the constant k.

(ii)evaluate P(X<6), P(X≥6), P(3<X≤6), and P(3< X/X ≤ 6)

(iii)find the minimum value of x so that P(X≤x)>1/2

(iv)obtain the distribution function F(x)

(i) Since *f(x)* is a p.m.f, $\Sigma_x f(x) = 1$

$$\therefore \sum_{x=0}^{7} f(x) = 1 \rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$= 10k^2 + 9k - 1 = 0$$

$$k = -1, \frac{1}{10}$$

$$\therefore k = \frac{1}{10} \quad [since\ f(x) \geq 0, \forall x = 0,1,2 \cdots 7\ and\ so\ k \neq -1]$$

(ii) $P(X < 6) = 1 - P(X \geq 6) = 1 - \{P(X = 6) + P(x = 7)\}$

$$= 1 - \left\{2\left(\frac{1}{10}\right)^2 + 7\left(\frac{1}{10}\right)^2 + \frac{1}{10}\right\} = \frac{81}{100}$$

$$\therefore P(X < 6) = 1 - P(X < 6) = 1 - \frac{81}{100} = \frac{19}{100}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{33}{100}$$

$$P < \frac{X}{X} \leq 6 = \frac{P\{(3 < X) \cap (X \leq 6)\}}{P(X \leq 6)} = \frac{P(3 < X \leq 6)}{P(X \leq 6)} = \frac{33/100}{83/100} = \frac{33}{83}$$

(iii) $P(\leq 3) = P\left(\frac{1}{10} < \frac{1}{2}\right), P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{1}{10} + 2 \cdot \frac{1}{10} = \frac{3}{10} < \frac{1}{2}$$

$$P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{10} + 2 \cdot \frac{1}{10} + 2 \cdot \frac{1}{10} = \frac{1}{2}$$

$$P(X \leq 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(x = 4)$$

$$= \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$$

Thus minimum value of x so that $P(X \leq x) > \frac{1}{2}$ is 4.

Lecture 14:

Binomial Distribution.

Definition. The **probability mass function of a binomial random variable *X*** is:

$$f(x) = {}^nC_x p^x (1-p)^{n-x}$$

We denote the **binomial distribution** as $b(n, p)$. That is, we say: $X \sim b(n, p)$

where the tilde (~) is read "as distributed as," and n and p are called **parameters** of the distribution.

- If X is a binomial random variable with **probability mass function**,

$$f(x)={}^nC_x p^x(1-p)^{n-x}$$

then the **mean of X** is: $\boldsymbol{\mu = np}$

Proof:

$$\begin{aligned}
\mu &= \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} \\
&= np\sum_{k=0}^{n} k\frac{(n-1)!}{(n-k)!k!}p^{k-1}(1-p)^{(n-1)-(k-1)} \\
&= np\sum_{k=1}^{n} \frac{(n-1)!}{((n-1)-(k-1))!(k-1)!}p^{k-1}(1-p)^{(n-1)-(k-1)} \\
&= np\sum_{k=1}^{n} \binom{n-1}{k-1}p^{k-1}(1-p)^{(n-1)-(k-1)} \\
&= np\sum_{\ell=0}^{n-1} \binom{n-1}{\ell}p^{\ell}(1-p)^{(n-1)-\ell} && \text{with } \ell := k-1 \\
&= np\sum_{\ell=0}^{m} \binom{m}{\ell}p^{\ell}(1-p)^{m-\ell} && \text{with } m := n-1 \\
&= np(p+(1-p))^m \\
&= np
\end{aligned}$$

- If X is a binomial random variable, then the **variance of X** is:

$$\sigma^2=np(1-p)$$

and the s**tandard deviation of X** is: $\boldsymbol{\sigma}=\sqrt{np(1-p)}$

Proof. Hintz.

The definition of the expected value of a function gives us:

$$E[X(X-1)]=\sum_{X=0}^{n} x(x-1)\times f(x)=\sum_{X=0}^{n} x(x-1)\,f(x)={}^nC_x p^x(1-p)^{n-x}$$

Left as an exercise.

Lecture 15:

Poisson Distribution.

Definition. The **probability mass function of a Poisson random variable** ***X*** is:

$$P(X = x) \quad = \quad e^{-\lambda}\frac{\lambda^x}{x!} \qquad x = 0, 1, 2, 3, 4, \ldots$$

We denote the **Poisson distribution** as $b(n, p)$. That is, we say:

$$X \sim P_0(\lambda)$$

where the tilde (~) is read "as distributed as," and λ is called **parameter** of the distribution.

Example:
Births in a hospital occur randomly at an average rate of 1.8 births per hour.
What is the probability of observing 4 births in a given hour at the hospital?

Let
X= No. of births in a given hour=4
(i) Events occur randomly
(ii) Mean rate λ= 1.8 $\Rightarrow X \sim Po(1.8)$
We can now use the formula to calculate the probability of observing exactly 4 births in a given hour

$$P(X=4) = e^{-1.8}\frac{1.8^4}{4!} = 0.072$$

What about the probability of observing more than or equal to 2 births in a given hour at the hospital?

We want $P(X \geq 2) = P(X = 2) + P(X = 3) + \ldots$

i.e. an infinite number of probabilities to calculate

but

$$\begin{aligned} P(X \geq 2) &= P(X=2) + P(X=3) + \ldots \\ &= 1 - P(X<2) \\ &= 1 - (P(X=0) + P(X=1)) \\ &= 1 - (e^{-1.8}\frac{1.8^0}{0!} + e^{-1.8}\frac{1.8^1}{1!}) \\ &= 1 - (0.16529 + 0.29753) \\ &= 0.537 \end{aligned}$$

- If X is a Poisson random variable with **probability mass function**,

$$\mathbf{P(X = x)} = e^{-\lambda}\frac{\lambda^x}{x!} \qquad x = 0, 1, 2, 3, 4, \ldots$$

t

Then the **mean (** μ **) of** X and **variance** (σ^2) of **X** are respectively λ.

Proof. Left as an exercise.

Some Problems:

1.

Given that 5% of a population are left-handed, use the Poisson distribution to estimate the probability that a random sample of 100 people contains 2 or more left-handed people.

X = No. of left handed people in a sample of 100

$X \sim \text{Bin}(100, 0.05)$

Poisson approximation $\Rightarrow X \sim \text{Po}(\lambda)$ with $\lambda = 100 \times 0.05 = 5$

We want $P(X \geq 2)$?

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - \Big(P(X = 0) + P(X = 1)\Big) \\ &\approx 1 - \left(e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!}\right) \\ &\approx 1 - 0.040428 \\ &\approx 0.959572 \end{aligned}$$

Lecture 16:

Normal Distribution.

The normal distribution is informally called the **bell curve**.

The probability density function of the normal distribution is:

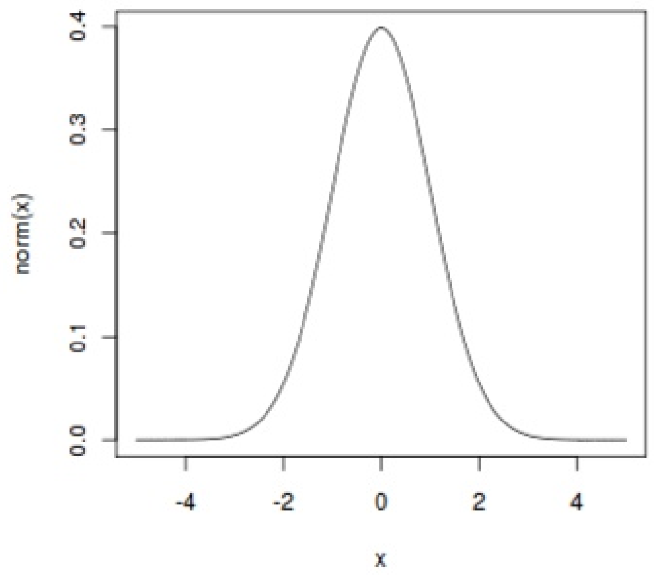
$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are respectively the expectation and the standard deviation of the distribution. A random variable with a Gaussian distribution is said to be **normally distributed** and is called a **normal deviate**.

If a random variable X follows the normal distribution, then we write:

$$X \sim N(\mu, \sigma^2)$$

In particular, the normal distribution with $\mu = 0$ and $\sigma = 1$ is called the ***standard normal distribution***, and is denoted as $N(0,1)$. It can be graphically represented as follows;



This is a special case when $\mu = 0$ and $\sigma = 1$, and it is described by this probability density function:

$$\varphi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

The distribution function of the standard normal distribution, usually denoted with the capital Greek letter Φ, is the integral

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

Lecture-17

Some Problems on Normal distribution.

1. If X follows a normal distribution with mean 12 and variance 16, find $P(X \geq 20)$.

Solution: Var(X)= 16

Therefore, standard deviation σ= 4

Let $Z = \frac{X-\mu}{\sigma} = \frac{X-12}{4}$

Therefore, $Z \sim N(0,1)$

When $X = 20,\ Z = 2$

Hence, $P(X \geq 20) = P(Z \geq 2)$

$$= 1 - P(Z < 2)$$

$$= 1 - \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = 1 - 0.977725 = 0.02275$$

2. If the weekly wage of 10,000 workers in factory follows normal distribution with mean wage Rs.70 and standard deviation Rs.5 respectively, find the number of workers whose weekly wage is between Rs. 66 and Rs.72.

Given: $\int_{-\infty}^{0.4} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = 0.1554$ and $\int_{-\infty}^{0.8} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = 0.2881$

Solution: let X be the random variable that corresponds to the weekly wage of a worker

Let, $= \frac{X-\mu}{\sigma}$, therefore $Z \sim N(0,1)$

When $X = 66$, $Z = \frac{X-\mu}{\sigma} = \frac{66-70}{5} = -0.8$

And when $= 72$, $\square = \frac{\square - \square}{\square} = \frac{72-70}{5} = 0.4$

Therefore, $\square(66 \leq \square \leq 72) = P(-0.8 \leq \square \leq 0.4) = \square(-0.8 \leq \square \leq 0) + \square(0 < \square \leq 0.4)$

$$= \square(0 \leq \square \leq 0.8) + \square(0 < \square \leq 0.4)$$

$$= 0.2881 + 0.1554 = 0.4435 = 0.2881 + 0.1554 = 0.4435$$

3. The mean weight of 1000 students in an engineering college is 65 kg and standard deviation is 5 kg. Find the number of students having weight between 55 kg and 70 kg. Assume the weight of the students follow normal distribution. Given $\square(2) = 0.9772$, $\square(1) = 0.8413$

Lecture-18

Combination of two independent random variables

If X_1 and X_2 are two independent standard normal random variables with mean 0 and variance 1, then their sum and difference is distributed normally with mean zero and variance two: $X_1 \pm X_2 \ \square\ N(0, 2)$.

Correlation Coefficient and Regression Lines:

A **correlation coefficient** is a number that quantifies a type of dependence and correlation, meaning statistical relationships between two or more values in fundamental statistics.

Correlation Coefficient: $r(or\ \rho) = \dfrac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}}{\sigma_x \sigma_y} = \dfrac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$

Regression line Y on X is $Y - \bar{y} = r\dfrac{\sigma_y}{\sigma_x}(X - \bar{x})$

Regression line X on Y is $X - \bar{x} = r\dfrac{\sigma_x}{\sigma_y}(Y - \bar{y})$

Example : If $r = 0.4, \operatorname{cov}(x, y) = 10, \sigma_y = 5,\ find\ \sigma_x.$

Ans.

$$r = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$$

$$0.4 = \frac{10}{\sigma_x X 5}$$

$$\sigma_x = 5$$

Example: Calculate the correlation coefficient and determine the regression lines of Y on X and X on Y for the sample

X	8	10	5	8	9
Y	1	3	1	2	3

Ans:
$$\bar{x} = \frac{8+10+5+8+9}{5} = 8$$

$$\bar{y} = \frac{1+3+1+2+3}{5} = 2$$

$$\sigma_x^{\ 2} = \frac{8^2 + 10^2 + 5^2 + 8^2 + 9^2}{5} - 8^2 = 2.8$$

$$\sigma_y^{\ 2} = \frac{1^2+3^2+1^2+2^2+3^2}{5} - 2^2 = 0.8$$

$$\frac{1}{5}\sum_{i=1}^{5} x_i y_i = \frac{1}{5}(8X1+10X3+5X1+8X2+9X3) = 17.2$$

$$r = \frac{\frac{1}{5}\sum_{i=1}^{5} x_i y_i - \overline{x}\,\overline{y}}{\sigma_x \sigma_y} = \frac{17.2-8X2}{\sqrt{2.8}\sqrt{0.8}} = 0.802$$

Therefore correlation coefficient =0.802

Regression line Y on X is

$$Y - \overline{y} = r\frac{\sigma_y}{\sigma_x}(X-\overline{x})$$

$$Y - 2 = 0.802\ x\ \sqrt{\frac{0.8}{2.8}}(X-8)$$

$$Y = 0.429X - 1.432$$

Regression line X on Y is

$$X - \overline{x} = r\frac{\sigma_x}{\sigma_y}(Y-\overline{y})$$

$$or,\ X - 8 = 0.802\ x\ \sqrt{\frac{2.8}{0.8}}(Y-2)$$

$$or,\ X = 1.5Y + 5$$

Lecture 19.

Curve Fitting

Least Squares Method:

The **least squares method** is a form of mathematical regression analysis that finds the line of best fit for a dataset, providing a visual demonstration of the relationship between the data points. The linear fit that matches the pattern of a set of paired data as closely as possible. Out of all possible linear fits, the **least-squares regression** line is the one that has the smallest possible value for the sum of the **squares** of the residuals.

Linear Curve Fitting

Example: Fit a linear equation to the following data

X	2	4	6	8	10
Y	2.3	2.6	2.9	2.10	2.12

Ans:

x_i	y_i	$x_i X\ y_i$	x_i^2
2	2.3	4.6	4
4	2.6	10.6	16
6	2.9	17.4	36
8	2.10	16.80	64
10	2.12	21.20	100
$\sum x_i = 30$	$\sum y_i = 12.02$	$\sum x_i X\ y_i = 70.4$	$\sum x_i^2 = 220$

Let the linear equation be $y = a + bx$ where a, b are constants.

We get the normal equations as

$$\sum y_i = na + b\sum x_i$$
$$\sum x_i X\ y_i = a\sum x_i + b\sum x_i^2$$

Where n is the number of elements.

Here n=5

From the table we get

$$12.02 = 5a + 30b$$
$$70.4 = 30a + 220b$$

Solving these two equations we get

$$a = 10.73$$
$$b = 0.043$$

Therefore the fitted linear curve is $y = 10.73 + 0.043x$.

Lecture 20.

Example: Fit a parabola $y = a + bx + cx^2$ using least square method for the following data

X	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Y	1.1	1.3	1.6	2.0	2.7	3.3	4.1

Ans:

x_i	y_i	x_i^2	$x_i X\ y_i$	x_i^3	$x_i^2 y_i$	x_i^4
1.0	1.1	1.0	1.1	1.0	1.1	1.0
1.5	1.3	2.25	1.95	3.375	2.925	5.0625
2.0	1.6	4.0	3.2	8	6.4	16.0
2.5	2.0	6.25	5.0	15.625	12.5	39.0625
3.0	2.7	9.0	8.1	27	24.3	81
3.5	3.3	12.25	11.55	42.875	40.425	150.0625
4.0	4.1	16.0	16.4	64	65.6	256
$\sum x_i = 1$	$\sum y_i = 16$	$\sum x_i^2 = 50.7$	$\sum x_i X\ y_i = 4$	$\sum x_i^3 = 161.$	$\sum x_i^2 y_i = 15$	$\sum x_i^4 = 548.$

Using least square method, we get the normal equations as

$$\sum y_i = na + b\sum x_i + c\sum x_i^2$$
$$\sum x_i X\ y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3$$
$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4$$

Where n is the number of observations.

Here n=7

From the above table we get,

$$16.1 = 7a + 17.5b + 50.75c$$
$$47.3 = 17.5a + 50.75b + 161.875c$$
$$153.25 = 50.75a + 161.875b + 548.1875c$$

Solving the above equations, using Gauss Elimination method, we get

$$a = 1.0571$$
$$b = -0.20714$$
$$c = 0.24286$$

Therefore the fitted parabola is $y = 1.0571 - 0.20714x + 0.24286x^2$

Some Problems (MCQ)

1. Four coins are tossed. Expectation of number of heads is
   (a) 1 (b)2
   (c) 3 (d)4
2. A card is drawn at random from a well shuffled pack of 52 cards. The probability of getting a heart or a diamond is
   (a) 1 **(b)1/2**
   (c) 1/26 (d)3/13

3. Let A, B be two events and P($\bar{A}$) = 0.3, P(B) =0.4, P(A $\bar{B}$) = 0.5, then P(A+ $\bar{B}$) =
   (a) 0.5 **(b)0.8**
   (c) 1 (d) None of these

4. The probability that a leap year, selected at random, will contain 53 Sunday is
   (a) 1/7 (b)2/365
   (c) 2/7 (d) None of these

5. If A and B are two mutually exclusive events, then P(A+B) =

(a) P(A)+P(B) (b) P(A) P(B)

(c) P(A) –P(B) (d) None of these

6. If P(A) = 1/2, P(B) = 1/3, P(AB) = ¼, then the value of $P(A\cup B)$ is

(a) 6/7 (b) 3/7

(c) 1 **(d) 7/12**

7. " Two mutually exclusive events are always independent ". This statement is

(a) true **(b) false**

8. If $\bar{A}$ is the complementary event of A, then

(a) P($\bar{A}$)=1-P(A) (b) P($\bar{A}$)=P(A)

(c) P($\bar{A}$)=1+P(A) (d) None of these

9. An unbiased die is thrown. The probability that either an even number or a number greater than 2 will turn up is

(a) 1/6 (b) 2/3

(c) 5/6 (d) None of these

10. A man draw at random three balls from a bag containing 6 red and 5 green balls. The probability of getting the balls all red is

(a) 6/11 (b) 3/22

(c) 4/33 (d) 1/6

MODULE III

Calculus of Complex Variable

Lecture 21.

COMPLEX POINT

A collection of points in the complex plane (Argand plane) is called a point set . So the complex point set is nothing but the set of some complex numbers. For example the set { 1+2i , -1+6i , 0+3i } is a complex point set.

Neighbourhood of a point.

Let z_o be a point of the complex plane. The set of all points z satisfying the inequality $|z - z_o| < \in$ is called the neighbourhood of z_o and is denoted by $N(z_o, \in)$. For example $N(1+2i, 0.3)$ is a neighbourhood of the region inside the circle with centre (1,2) and Radius 0.3 excluding the points on the circumference.

If from the neighbourhood of a point $z_o$we exclude the point $z_o$itself then such a neighbourh hood is called the deleted neighbourhood of z_o. and is represented by $0 < |z - z_o| < \in$ and is denoted by $N^o(z_o, \in)$.

Limit Point.

A point z_o is called a limit point of a set S if every neighbourhood$N(z_o)$ contain at least one point of S other than z_o.

FuntionOf A Complex Variable

When a symbol z takes any one of the values of a set of complex numbers then z is called a complex variable.

Let D and R be any two non empty point sets in the complex plane. A complex variable $w \in R$ is said to be the function of a complex variable $z \in D$,if to every value of z corresponds oneor valus of w. Thus if w is a function of z , it is written as w = f(z).

If z = x+iy and w = u + iv then u and v are both functions of real variable and we may Write w = f(z) = u(x,y) + i v(x,y) . Here D is called Domain and R is called range of f.

Illustration 1. Consider the function $f : S \to C$, given by $f(z) = z^2$ and where $S = \{z \in C : |z| < 2\}$ is the open disc with radius 2 and centre 0.

Using polar coordinates, it is easy to see that the range of the function is the open disc $f(S) = \{w \in C : |w| < 4\}$ with radius 4 and centre 0.

Limits and Continuity.

The concept of a limit in complex analysis is exactly the same as in real analysis.

So, for example, we say that $f(z) \to L$ as $z \to z_o$, or

$$\lim_{z \to z_o} f(z) = L$$

if, given any $\in > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \in$ whenever $0 < |z - z_o| < \delta$.

Similarly, we say that a function f(z) is continuous at z_o if $f(z) \to f(z_o)$ as $z \to z_o$. A similar qualification on z applies if z_o is a boundary point of the region S of definition of the function. We also say that a function is continuous in a region if it is continuous at every point of the region.

Note that for a function to be continuous in a region, it is enough to have continuity at every point of the region. Hence the choice of δ may depend on a point z_o in question. If δ can be chosen independently of z_o, then we have some uniformity as well.

Theorems on Continuity.

Theorem 1. A necessary and sufficient condition of a function $f(z) = u + iv$ to be continuous At $z_0 = x_0 + iy_0$ is that u(x,y) and v(x,y) be continuous at (x_0 , y_0).

Theorem 2. If the function f(z) and g(z) are defined in D and continuous at $z = z_0$, then

(i) pf(z) +qg(z) (p , q are constants)
(ii) f(z)g(z)
(iii) f(z)/g(z) if $g(z) \neq 0$

are also continuous at $z = z_0$.

Example 1.Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Solution.Let $z = x+iy$ then $\bar{z} = x-iy$ therefore $\frac{\bar{z}}{z} = \frac{x - iy}{x + iy}$.

Let $z \to 0$ along x axis . Then y=0 thus $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = 1$.

Next let $z \to 0$ along y axis then x=0

Therefore $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{-iy}{y} = -1$.

Thus $\lim_{z\to 0}\frac{\bar{z}}{z}$ has different values along different path.

Hence $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

Example 2. Test the continuity at origin of the following function

$$f(z) = \frac{xy^3}{x^2+y^6} \quad \text{for } z \neq 0$$
$$= 0 \quad \text{for } z = 0$$

Solution. Let $f(z) = u(x,y) + iv(x,y)$, Then

$$u(x,y) = \frac{xy^3}{x^2+y^6} \quad , (x,y) \neq (0,0)$$
$$= 0 \quad (x,y) = (0,0)$$

$v(x,y) = 0 \quad \forall (x,y)$

Let $(x,y) \to (0,0)$ along the curve $x=my^3$.

Therefore, $\lim_{(x,y)\to(0,0)} u(x,y) = \lim_{y\to 0}\frac{my^3y^3}{m^2y^6+y^6} = \frac{m}{1+m^2}$

Thus, $\lim_{(x,y)\to(0,0)} u(x,y)$ has different values for different values of m

Hence , $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist . Therefore u(x,y) is not continuous at (0,0). Consequently f(z) is not continuous at z= 0.

Lecture 22.

Differentiability

A function w = f(z) defined in a certain domain D is said to be differentiable at $z = z_o$ if the

Limit $\lim_{z\to z_o}\frac{f(z)-f(z_o)}{z-z_o}$ exists

Theorem 1. If f(z) = u(x,y) + iv(x,y) is differentiable then u and v are also differentiable .

Moreover, $f'(z) = u_x + iv_x$ and $f'(z) = -iu_x + v_y$.

Proof. $f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$.

Here $\Delta z \to 0$ along any path . Let $\Delta z = h \to 0$ along real axis . Then

$$f'(z) = \lim_{\Delta Z\to 0}\frac{f(z+h)-f(z)}{h}$$

$$= \lim_{h\to 0}\frac{\{u(x+h,y)+iv(x+h,y)\}-\{u(x,y)+iv(x,y)\}}{h}$$

$$= \lim_{h\to 0}\left[\frac{\{u(x+h,y)-u(x,y)\}}{h}+i\frac{v(x+h,y)-v(x,y)}{h}\right] = u_x + iv_x$$

Next let $\Delta z \to 0$ along imaginary axis i.e, $\Delta z = ik$ and $k \to 0$

Then, $f^{/}(z) = \lim_{k\to 0}\frac{f(z+ik)-f(z)}{ik}$

$$= \lim_{k\to 0}\frac{\{u(x,y+k)+iv(x,y+k)\}-\{u(x,y)+iv(x,y)\}}{ik}$$

$$= \lim_{k\to 0}\frac{\{u(x,y+k)-u(x,y)\}}{ik}+i\lim_{k\to 0}\frac{v(x,y+k)-v(x,y)}{ik}$$

$$= -I\lim_{k\to 0}\frac{\{u(x,y+k)-u(x,y)\}}{k}+\lim_{k\to 0}\frac{v(x,y+k)-v(x,y)}{k} = -iu_y+v_y$$

Theorem 2. If a function is differentiable at a point , then it is continuous at that point.
Proof. Left as an exercise

<u>Analytic Functions</u>

If a function f(z) be such that $f^{/}(z)$ exists at every point of the domain D then f(z) is said to be analytic in D

<u>Cauchy Riemann conditions</u>

The necessary conditions for w = f(z) = u(x,y) + iv(x,y) is analytic at any point z = x+iy

Of its domain D is that the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ should exist and

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Example3. For the function defined by f(z) = $\sqrt{|xy|}$ show that the Cauchy Riemann Equation are satisfied at (0,0) but the function is not differentiable and analytic at that Point.

<u>Solution.</u> Let f(z) = u(x,y) + iv(x,y).

Then u(x,y) = $\sqrt{|xy|}$ and v(x,y) = 0

Now at the origin,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{x\to 0}\frac{0-0}{y} = 0$$

Similarly $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

Hence Cauchy Riemann equation is satisfied at origin.

Again $f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{(x,y)\to 0}\frac{\sqrt{|xy|}}{x+iy}$.

Let $z \to 0$ along the straight line y=mx. Then

$f'(0) = \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x+imx} = \frac{\sqrt{|m|}}{1+im}$ which have different values for different m. Hence

$f'(0)$ does not exist. Thus the function is not differentiable at the origin.

Lecture 23.

Laplace's Equation

A partial differential equation of the form $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$=0 is called Laplace's Equation

Harmonic Function

A function f(x,y) which possesses continuous partial derivatives of first and second orders And satisfies Laplace Equation is called Harmonic function.

Conjugate Harmonic Function

If the two harmonic functions u(x,y) and v(x,y) satisfy the Cauchy Riemann equations then they are called Conjugate harmonic functions

Construction of analytic functions

Milne Thomson Method:

Letf(z) = u+iv be an analytic function where u, v are conjugate harmonic . If one these Say u is given then determination of f(z) directly without finding v is due to milne Thomson

Example 4. Prove that u = x^3-3xy+3x^2-3y^2+1 is a harmonic function and determine the corresponding analytic function u+iv.

Solution. Here u = x^3-3xy+3x^2-3y^2+1 then $\frac{\partial u}{\partial x}$ = 3x^2-3y^2+6x and $\frac{\partial^2 u}{\partial x^2}$ = 6x+6

$\frac{\partial u}{\partial y}$ = -6xy-6y, $\frac{\partial^2 u}{\partial y^2}$ = -6x-6

Thus $\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$ and hence u is a harmonic function.

Let $\frac{\partial u}{\partial x} = \phi_1(x,y)$, $\frac{\partial u}{\partial y} = \phi_2(x,y)$

$\phi_1(x,y)$ = 3x^2-3y^2+6x and $\phi_2(x,y)$ = -6xy-6y

Therefore By milne's method we have
$f'(z) = \phi_1(z,0) - i\phi_2(z,0) = (3z^2+6z) - i.0$,
Integrating we get $f(z) = z^3+3z^2+c$ where c is an arbitrary constant.

Lecture 24.

COMPLEX INTEGRATION

Curve represented by a complex variable

Let z = x+iy be a complex variable. Since a fixed z represents a point (x,y) in Argand Plane, so as z varies the point (x,y) moves on the plane and makes a locus or a curve. We say this curve, say C is represented by the complex variable z.

Parametric Representation

Let z = x+iy be a complex variable. If $x = \phi(t)$ and $y = \psi(t)$
Where t is a real variable then there is a relation between x and y . This relation gives a Curve , C (say)
Then we say $z = \phi(t) + i\psi(t)$ gives the curve C , t is a parameter.

Simple Curve.

A curve C is called simple if it does not intersect itself. So a curve $C : z = \phi(t) + i\psi(t)$
Is simple if $t_1 \neq t_2$ implies $z(t_1) \neq z(t_2)$.

Closed Curve.

A simple curve is called closed if the two end points of the curve coincide

Smooth Curve.

A curve C is called smooth if it possess unique tangent at every point.

Contour or Piecewise smooth curve.

A curve is called contour or piecewise smooth if it is comprised of a finite number of smooth curves.

Cauchy's Theorem.

Let f(z) be an analytic function and $f'(z)$ is continuous at each point within the domain D bounded by a closed contour C. Then,

$$\oint_c f(z)dz = 0$$

Proof. Let f(z) = u(x,y) + iv(x,y).

As f(z) is an analytic function , so by Cauchy Riemann condition,

We have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also $f'(z)$ is continuous and as $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

So u, v and their partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous within and on C.

So Greens theorem can be applied. $\oint_c f(z)dz = \oint_c (u+iv)(dx+idy)$

$$= \oint_c (udx - vdy) + i(vdx + udy) = \oint_c (udx - vdy) + i\oint_c (vdx + udy)$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \quad \text{, by Greens Theorem}$$

$$= \iint_D \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy$$

= 0 +i.0

Lecture 25.

Cauchy Goursat Theorem.

Let f(z) be analytic function within and on a simple closed contour C. Then

$\oint_C f(z)dz = 0$

Formula 1. $\oint_C \frac{dz}{z-\alpha} = 0$ if C is any simple closed curve and $z = \alpha$ is an exterior point of C

As $z = \alpha$ is an exterior point of C so $f(z) = \frac{1}{z-\alpha}$ is an analytic function

Everywhere within and on C. Hence by Cauchy's theorem $\oint_C f(z)dz = 0$

i.e, $\oint_C \frac{dz}{z-\alpha} = 0$.

Formula 2. $\oint_C \frac{dz}{z-\alpha} = 2\pi i$, if C is any simple closed curve and $z = \alpha$ is an interior point of C

Formula 3. $\oint_C \frac{dz}{(z-\alpha)^n} = 0$, n = 2,3,4,……..

Where α is an interior point of any simple closed curve C.
As in formula 1 we can write

$\oint_C \frac{dz}{(z-\alpha)^n} = \oint_{C_1} \frac{dz}{(z-\alpha)^n}$ where C_1 is a circle lying within C and the equation

Of the circle C_1 is $z - \alpha = \rho$

On the circle $|z-\alpha| = \rho$,

We have, $z-\alpha=\rho(\cos\theta+i\sin\theta)=\rho e^{i\theta}$

i.e, $z=\alpha+\rho e^{i\theta}$ where θ varies from 0 to 2π

Therefore $dz=\rho i e^{i\theta}d\theta$.

Thus $\oint_{C_1}\frac{dz}{(z-\alpha)^n}=\int_0^{2\pi}\frac{i\rho e^{i\theta}}{\rho^n e^{in\theta}}=\frac{i}{\rho^{n-1}}\left[\frac{e^{(1-n)i\theta}}{(1-n)}\right]_0^{2\pi}=0.$

Hence, $\oint_C\frac{dz}{(z-\alpha)^n}=0 \qquad n=2,3,\ldots\ldots.$

Lecture 26.

Cauchy Integral Formula

Theorem 1. If f(z) is analytic within and on a simple closed curve C and α is any point within C. Then $f(\alpha)=\frac{1}{2\pi 1}\oint_C\frac{f(z)}{z-\alpha}dz$.

Proof. As $\frac{f(z)}{z-\alpha}$ is analytic everywhere within C except at $z=\alpha$, we draw a circle C_1 With centre at $z=\alpha$ and radius r so that C_1 lies wholly within C. Then $\frac{f(z)}{z-\alpha}$ is analytic Within the annular region bounded by C and C_1 .Hence ,

$\oint_C\frac{f(z)}{z-\alpha}dz=\oint_{C_1}\frac{f(z)}{z-\alpha}dz$. Now the equation of the circle C_1 is $|z-\alpha|=r$ thus

$z-\alpha=re^{i\theta}$ where θ varies from 0 to 2π

Therefore $dz=rie^{i\theta}d\theta$

Thus $\oint_{C_1}\frac{f(z)}{z-\alpha}dz=\int_0^{2\pi}\frac{f(\alpha+re^{i\theta})}{re^{i\theta}}rie^{i\theta}d\theta$

Now taking limit as $r\to 0$ on both sides of above equation we have,

$$\oint_{C_1}\frac{f(z)}{z-\alpha}dz = i\lim_{r\to 0}\int_0^{2\pi}f(\alpha+re^{i\theta})d\theta = if(\alpha)\int_0^{2\pi}d\theta=2\pi i f(\alpha)$$

Hence $f(\alpha)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-\alpha}dz$.

NOTE : When α is an exterior point of C then $\frac{f(z)}{z-\alpha}$ is analytic within and on C.

So by Cauchy's theorem $\oint_C \frac{f(z)}{z-\alpha}dz = 0$

Lecture 27.

Cauchy Integral Formula

Theorem 2. If f(z) is analytic within and on a closed curve C , then the derivative of f(z) at an Interior point α of C is given by $f'(\alpha) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^2}\,dz$.

Theorem 3. If f(z) is analytic within and on a closed curve C , then the nth order derivative of f(z) at any interior point α of C is given by

$$f^n(\alpha) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^{n+1}}dz.$$

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate $\oint_C \frac{e^z}{(z+1)(z+2)}dz$ Where C is the circle $|z-1| = 4$.

Solution. Here f(z) = e^z is analytic within and on the circle $|z-1| = 4$ and z = -1, -2 are the interior

Points of C. Now $\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$.

Therefore $\oint_C \frac{e^z}{(z+1)(z+2)}dz = \oint_C \frac{e^z}{(z+1)}dz - \oint_C \frac{e^z}{(z+2)}dz$

= $2\pi i \times e^{-1} - 2\pi i \times e^{-2}$, by Cauchy's integral formula

= $2\pi i(e^{-1} - e^{-2})$.

Example2. Evaluate $\oint_C \frac{\sin 3z + 2\cos z}{(z+\frac{\pi}{2})}dz$ if C is the circle $|z| = 5$

Solution. Here f(z) = sin3z +2cosz is analytic within and on the circle $|z| = 5$ and $z = \frac{-\pi}{2}$ lies Inside the circle C. Hence by Cauchy integral formula

$$\oint_C \frac{\sin 3z + 2\cos z}{(z+\frac{\pi}{2})}dz = 2\pi i(\sin 3(\frac{-\pi}{2}) + 2\cos(\frac{-\pi}{2})) = 2\pi i(1+2.0) = 2\pi i.$$

Example 3. Evaluate $\oint_C \frac{\cos^3 z}{(z-\frac{\pi}{4})^3}dz$ where C is the circle $|z|=1$.

Solution. Let f(z) = $\cos^3 z$ which is analytic within and on C . Also $z=\frac{\pi}{4}$ lies inside the circle C .

Hence by Cauchy's integral formula $f''(\alpha)=\frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha)^3}dz$.

We get $f''(\frac{\pi}{4})=\frac{1}{\pi i}\oint_C \frac{\cos^3 z}{(z-\frac{\pi}{4})^3}dz$.

Now $\oint_C \frac{\cos^3 z}{(z-\frac{\pi}{4})^3}dz = \pi i(6\cos\frac{\pi}{4}\sin^2\frac{\pi}{4}-3\cos^3\frac{\pi}{4})$

$$= \frac{3\sqrt{2}\pi i}{4}.$$

Lecture 28

Taylor's Theorem

Let f(z) be analytic at all points within a circle C_0 with centre z_0 and radius r_0. Then for every point z within C_0 , we have

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+.....+\frac{f^n(z_0)}{n!}(z-z_0)^n+...$$

$$= f(z_0)+\sum_{n=1}^{\infty}\frac{(z-z_0)^n}{n!}f^n(z_0).$$

If we put $z_0 = 0$ in the above series we get ,

$f(z)=f(0)+\sum_{n=1}^{\infty}\frac{(z)^n}{n!}f^n(0)$, which is known as Maclaurin's series

Example 1. Find the Taylor's expansion of $f(z)=\frac{1}{(z+1)^2}$ about the point z = -i

Solution. To expand f(z) about z = -i in powers of z+i put z+i = t . Then

$$f(z)=\frac{1}{(t-i+1)^2}=(1-i)^{-2}[1+\frac{t}{1-i}]^{-2}=\frac{i}{2}\left[1-\frac{2t}{1-i}+\frac{3t^2}{(1-i)^2}-\frac{4t^3}{(1-i)^3}+\dots\right]$$

$$=\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

Laurent's Series.

If a function f(z) is analytic in the annulus (ring shaped) region D bounded by two concentric circles C_1 and C_2 with the centre at the point $z=\alpha$ and radius r_1 and r_2. ($r_1>r_2$), then for all z In D ,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{n=1}^{\infty}b_n(z-\alpha)^{-n}.$$

Where $a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^{n+1}}dz$, n= 0,1,2…

And $b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-\alpha)^{-n+1}}dz$, n= 1,2,…

Example 1. Expand the function f(z) = $\frac{z^2-1}{(z+2)(z+3)}$ when (i) $|z|<2$ (ii) $2<|z|<3$ and (iii) $|z|>3$

Solution. Let f(z) = 1+ $\frac{A}{z+2}+\frac{B}{z+3}$.

Then $z^2-1=(z+2)(z+3)+A(z+3)+B(z+2)$.

Putting z = -2 , -3 we get A = 3, B =-8. Therefore f(z) = 1 + $\frac{3}{z+2}-\frac{8}{z+3}$.

When $|z|<2$,

$$f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\dots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\dots\right)$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n(\frac{z}{2})^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n(\frac{z}{3})^n.$$

Example 2.Find the Laurent's series of the function $\frac{e^z}{(z-2)^3}$ about the point z=2.

Solution. Let z-2 = u then z = u +2.

Therefore, $\dfrac{e^z}{(z-2)^3} = \dfrac{e^{u+2}}{(u)^3} = \dfrac{e^2}{(u)^3}\left(1+u+\dfrac{u^2}{2!}+\dfrac{u^3}{3!}+\dots\right)$

$= e^2\left(\dfrac{1}{u^3}+\dfrac{1}{u^2}+\dfrac{1}{2!}\dfrac{1}{u}+\dfrac{1}{3!}+.\dfrac{1}{4!}u+\dots..\right)$

$= \dfrac{e^2}{(z-2)^3}+\dfrac{e^2}{(z-2)^2}+\dfrac{1}{2!}\dfrac{1}{z-2}+\dfrac{1}{3!}+\dfrac{1}{4!}(z-2)+\dots$

Lecture 29

Zero and Singularities of an Analytic Function.

Definition. A point z = a is said to be a zero of an analytic function f(z) if f(a) = 0.

Order of zero. If f(z) is analytic in domain D and $a \in D$ then a is called zero of f(z) of

Order m if $f(a) = f^{/}(a) = f^{//}(a) = \dots\dots\dots = f^{m-1}(a) = 0$ but $f^{m}(a) \neq 0$.

Thus from Taylors theorem f(z) = $\sum_{n=m}^{\infty} a_n (z-a)^n$ if a is mth order 0 of f(z) .

Singularities of an Analytic function

If a function f(z) is not analytic at the point z = a, then a is called the singularity or singular Point of f(z) .

Isolated and Non isolated singularity.

A singularity z = a of a function f(z) is said to be an isolated singularity if there is no other Singularity within a small neighbourhood of z = a.

If a singularity z = a of a function f(z) is not isolated then it is called non isolated singularity.

ILLUSTRATION

1. The function f(z) = $\dfrac{1}{z-1}$ is analytic everywhere except at z = 1. So z = 1 is the only

   Singularity of f(z). As the function f(z) has no other singularity, so z=1 is an isolated singularity.

2. The function f(z) = $\cot\dfrac{\pi}{z} = \dfrac{\cos\frac{\pi}{z}}{\sin\frac{\pi}{z}}$ is not analytic at z = 0 and at the points where $\sin\dfrac{\pi}{z} = 0$

   i.e. $\dfrac{\pi}{z} = n\pi$, n = 0,1,-1,2,-2,3,-3,…. I.e. $z = \dfrac{1}{n}$, n = 1,2,3,….

Essential Singularity and Pole

Principal Part of an Analytic Function.

Let α be an isolated singularity of an analytic function f(z) in a domain D. Now we draw a circle C_1 with center at z = α and radius as small as we please and another large concentric circle C_2 if any radius lying wholly within D. Then f(z) is analytic within the annular region between these two circles. Hence by Laurent's theorem, we have ,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} b_n (z-\alpha)^{-n} .$$

The second term in RHS is called the principal part of f(z) at the point z = α.

Essential Singularities.

If the principal part of f(z) (at the isolated singularity α) contains an infinite no of terms, then The singularity z = α is called an essential singularity.

Pole

If the principal part of f(z) contains a finite number of terms say m, then the singularity z = α Is called a pole of order m. In this case the Laurent's series of f(z) is of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \frac{b_1}{(z-\alpha)} + \frac{b_2}{(z-\alpha)^2} + \ldots\ldots + \frac{b_m}{(z-\alpha)^m} \;,\; b_m \neq 0$$

When m = 1 pole is said to be a simple pole.

ILLUSTRATION.

1. As f(z) = $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots$ contains an infinite number terms in negative power of z,

   So z = 0 is an essential singularity.

2. The function f(z) = $\frac{\sin(z-a)}{(z-a)^4} = \frac{1}{(z-a)^4}\left[(z-a) - \frac{1}{3!}(z-a)^3 + \frac{1}{5!}(z-a)^5 - \ldots\ldots\right]$

   $= \frac{1}{(z-a)^3} - \frac{1}{3!}(z-a) + \frac{1}{5!}(z-a) - \frac{1}{7!}(z-a)^3 + \ldots\ldots$ contains only two terms in negative power of z-a, so z = a is a pole of order 2.

Theorem 1. If an analytic function f(z) has a pole of order m at z = a then 1/f(z) has a Zero of order m at z = a.

Theorem 2. The limit point of the set of all poles of a function f(z) is a non isolated Essential singularity.

Theorem 3. The limit point of zeroes of a function f(z) is an isolated essential singularity.

ILLUSTRATION

Example 1. Find out the zeroes and discuss the nature of the singularities of

F(z) = $\frac{z-2}{z^2}\sin\frac{1}{z-1}$

Solution. Poles of f(z) are given by putting the denominator equal to zero i.e. $z^2 = 0$

So z = 0 is a pole of order 2.

Again the zeroes of f(z) are given by equating to zero the numerator of f(z)

i.e. $(z-2)\sin\frac{1}{z-1}=0$ or $z-2=0$ and $\sin\frac{1}{z-1}=0$

Now $z=2$ and $\sin\frac{1}{z-1}=0$ gives $\frac{1}{z-1}=n\pi$ i.e. $z=1+\frac{1}{n\pi}$

Thus all zeroes of f(z) are given by $z = 2,\ 1+\frac{1}{n\pi}$ where n = 1,-1,2,-2,3,-3…..

Also the limit point of zeroes given by $z = 1+\frac{1}{n\pi}$ (n = 1,2,3,….) is z =1 . Hence z =1 is an isolated essential singularity.

Example 2. Find the Laurent series about the indicated singularity for the function

$$F(z)=\frac{e^{z^2}}{z^3},\ z=0$$

Solution
$$f(z)=\frac{e^{z^2}}{z^3}=\frac{1}{(z)^3}\left[1+z^2+\frac{(z^2)^2}{2!}+\frac{(z^2)^3}{3!}.....\right]$$

$$=\frac{1}{z^3}+\frac{1}{z}+\frac{z}{2!}+\frac{z^3}{3!}.......$$

As the highest power of $\frac{1}{z}$ in the Laurent series is 3 , so z= 0 is a pole of order 3.

Lecture 30

Residue Theorem

Let α be an isolated singularity of an analytic function f(z). Then by Laurents Theorem ,

We have
$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{n=1}^{\infty}b_n(z-\alpha)^{-n}.$$

Where
$$a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-\alpha)^{n+1}}dz,\ n=0,1,2\ldots$$

And
$$b_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{(z-\alpha)^{-n+1}}dz,\ n=1,2,\ldots$$

The coefficient b_1 of $\frac{1}{(z-\alpha)}$ in the above infinite series is called the residue of f(z) at the Singularity $z=\alpha$ and is denoted by $\text{Res}(\alpha)$.

ILLUSTRATION

Consider the function $f(z)=\frac{1}{(z+1)^2(z-2)}$. Here $z=-1,2$ are the singularities of f(z)

Now $f(z)=\frac{1}{t^2(t-1-2)}$ by putting $z+1=t$.

$$= -\frac{1}{3t^2}\left(1-\frac{t}{3}\right)^{-1}$$

$$= -\frac{1}{3t^2}\left(1+\frac{t}{3}+\frac{t^2}{9}+...\right)$$

$$= -\frac{1}{3t^2} - \frac{1}{9}\frac{1}{t} - \frac{1}{27} - .....$$

$$= -\frac{1}{3(z+1)^2} - \frac{1}{9}\frac{1}{(z+1)} - \frac{1}{27} - ....$$

So the coefficient of $\frac{1}{(z+1)}$ is $-\frac{1}{9}$

Hence the residue at z = -1 is $-\frac{1}{9}$

Theorem. Let α be a pole of f(z) of order m . Then the residue of f(z) at z = α is given by

$$\frac{1}{(m-1)!}\lim_{z\to\infty}\frac{d^{m-1}}{dz^{m-1}}\left[(z-\alpha)^m f(z)\right]$$

Cauchy Residue Theorem

Let f(z) be analytic within and on a closed contour C except at a finite number of Singularities a_1 , a_2,…a_n and let R_1 , R_2,…R_n be respectively the residues of f(z) at these poles. Then , $\oint_c f(z)dz = 2\pi i(R_1 + R_2 + ...R_n)$.

Illustration

Ex 1. Use Cauchy Residue theorem to prove that

$$\oint_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}dz = -2\pi i \text{ where C is the circle } |z-1| = 1$$

Sol. Let f(z) = $\dfrac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}$

The Poles of f(z) is given by $\left(z-\frac{\pi}{2}\right)^3 = 0$ i.e. $z = \frac{\pi}{2}$ which is a pole of order 3 and lies within the circle $|z-1| = 1$. Hence by Cauchy's residue theorem we have $\oint_C f(z)dz = 2\pi i \operatorname{Res}\left(\frac{\pi}{2}\right)$

$$\text{Now } \operatorname{Res}\left(\frac{\pi}{2}\right) = \frac{1}{(3-1)!}\lim_{z\to\frac{\pi}{2}}\frac{d^2}{dz^2}\left[(z-\frac{\pi}{2})^3 f(z)\right]$$

$$= \frac{1}{2}\lim_{z\to\frac{\pi}{2}}(-2\sin z - z\cos z) = -1$$

Therefore $\oint_C f(z)dz = 2\pi i(-1)$

Hence $\oint_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}dz = -2\pi i$.

Ex 2. Evaluate $\oint_C \frac{z+1}{z^2-2z}dz$ where C is the circle $|z| = 5$

Sol. Left as an exercise.

Lecture 31

CONFORMAL MAPPING

Transformation or Mapping

Consider the complex valued function $w = f(z)$. Then corresponding to each point z_0, we have a point w_0 in the complex plane such that $f(z_0) = w_0$. Let $z = x + iy$ and $w = f(z) = u + iv$. Then $u = u(x,y)$, $v = v(x,y)$ represent a mapping which establish a correspondence between the points (x,y) in the x-y plane i.e. z plane and the points (u,v) in the uv-plane i.e.w plane. Here using the above mapping, a curve or a region of z plane transformed into another curve or a region of w plane.

IsogonalMapping: A mapping is said to be isogonal if two curves in the z plane intersecting at the point z_0 at an angle θ are transformed into two corresponding curves in the w-plane intersecting at the point w_0 which corresponds to z_0 at the same angle θ under the mapping.

Conformal Mapping : An isogonal mapping $w = f(z) = u + iv$ is said to be conformal if The sense of rotation as well as the magnitude of the angle is preserved

Theorem 1. If a function f(z) is analytic and $f'(z) \neq 0$ in a region D of the z plane then the mapping $w = f(z)$ is conformal at all points of D.

Illustration. Consider the function $f(z) = e^z$. Here e^z is analytic everywhere in the finite z plane and $f'(z) \neq 0$ for all z, so the mapping $w = e^z$ is conformal everywhere in the finite z plane.

Some special Transformation

Lecture 32

The Bilinear Transformation or Mobius Transformation

A transformation of the form $w = \frac{az+b}{cz+d}$ where a, b, c, d are constants and ad-bc $\neq 0$. Is Called bilinear transformation. The transformation can also be written as $cwz - az + wd - b = 0$

Which is linear both in w and z and hence the name bilinear. Now $\frac{dw}{dz} = \frac{ad - bc}{(cz+d)^2} \neq 0$

Everywhere except at z = ∞ and f(z) = $\frac{az+b}{cz+d}$ is analytic except when cz + d = 0 i.e. z = $-\frac{d}{c}$.

So the transformation is conformal everywhere in the finite z plane except at z = $-\frac{d}{c}$.

Illustration.

The transformation w = $\frac{2z-5}{z+4}$ is bilinear as 2x4-1x(-5) = 13 $\neq 0$ Now $\frac{dw}{dz} = \frac{13}{(z+4)^2} \neq 0$

Everwhere except at z = ∞ and f(z) = $\frac{2z-5}{z+4}$ is analytic everywhere except at z = -4. So the transformation is conformal everywhere in finite z plane except at z = -4.

Fixed Point of a Bilinear Transformation

When a point coincides with its image under a bilinear transformation, the point is called A fixed point or an invariant point.

Theorem1: A bilinear transformation having exactly one fixed point is of the form

$\frac{1}{w-p} = \frac{1}{z-p} + k$ where k $\neq 0$ and p is the fixed point.

This transformation is also known as parabola transformation.

Theorem 2 : A bilinear transformation having exactly two fixed points is of the form

$\frac{w-p}{w-q} = k\frac{z-p}{z-q}$ where k $\neq 0$ 1, p and q are fixed points

This transformation is also called elliptic if $|k| = 1$ and hyperbolic if k is real.

The above two form are known as normal or canonical form of bilinear transformation.

Illustaration

Ex 1. Consider the transformation w = $\frac{3z-4}{z-1}$

For a fixed point we have w = z .Therefore z = $\frac{3z-4}{z-1}$

$Z^2 - z = 3z - 4$ implies $(z-2)^2 = 0$ thus z = 2.

So z = 2 is the only fixed point.

Now $w - 2 = \frac{3z-4}{z-1} - 2 = \frac{z-2}{z-1}$ Thus ,

Thus $\frac{1}{w-2} = \frac{z-1}{z-2} = \frac{1}{z-2} + 1$ which is the normal form of the transformation and is called parabolic.

Theorem 3. The bilinear transformation which maps the points z_1, z_2, z_3 of the z plane into the points w_1 , w_2, w_3 of the w plane respectively is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Illustration :

The bilinear transformation which maps the points z = 2, I, -2. Into the points w =1, I , -1. Is given by

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

$$\frac{(w-1)(i+1)^2}{(w+1)(i^2-1)} = \frac{(z-2)(i+2)^2}{(z+2)(i^2-4)}$$

$\frac{(w-1)}{(w+1)}(-i) = \frac{(z-2)}{(z+2)} x \frac{3i-4}{-5}$ Using componendo and dividendo we get

$w = \frac{3z+2i}{iz+6}$.

Theorem 4: The bilinear transformation $w = \frac{az+b}{cz+d}$ where a, b, c, d are realconstants and ad-bc> 0 maps the upper half of the z plane into the upper half of w plane and conversely.

Ex. 1 Show that the line $y = \frac{x}{3}$ is mapped onto the circle under the bilinear transformation $W = \frac{(iz+2)}{(4z+i)}$. Find the centre and the radius of the image circle.

Soln. The transformation $W = \frac{(iz+2)}{(4z+i)}$. Can be written as $z = \frac{2-iw}{4w-i}$.

Putting z = x +iy , w = u +iv we get

$$x + iy = \frac{2-iu+v}{4u+4iv-i} = \frac{(v+2-iu)\{4u+i(4v-1)\}}{(4u)^2+(4v-1)^2}$$

$$= \frac{9u-i(4u^2+4v^2+7v-2)}{16u^2+(4v-1)^2}$$

Therefore $x = \frac{9u}{16u^2+(4v-1)^2}$, $y = \frac{-(4u^2+4v^2+7v-2)}{16u^2+(4v-1)^2}$

So the line $y = \frac{x}{3} i$ corresponds to $\frac{-(4u^2+4v^2+7v-2)}{16u^2+(4v-1)^2} = \frac{1}{3}\frac{9u}{16u^2+(4v-1)^2}$

i.e. $u^2+v^2+\frac{3}{4}u+\frac{7}{4}v-\frac{1}{2}=0$ which is the equation of the circle with centre $\left(\frac{-3}{8},\frac{-7}{8}\right)$ and radius $3\frac{\sqrt{10}}{8}$ in uv plane .

Ex 2. Find the bilinear transformation which maps z = i, 1, -1 onto w = 1, 0, ∞ respectively. Also Show that the unit circle $|z| = 1$ in z plane maps into the real axis of w plane.

Soln. Let the bilinear transformation be $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$.

Where $z_1 = I$, $z_2 = 1$, $z_3 = -1$, $w_1 = 1$, $w_2 = 0$, $w_3 = \infty$

As $w_3 = \infty$ so formula can be written as $$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

i.e. $$\frac{(w-1)(0-1)}{(0-1)(0-1)} = \frac{(z-i)(1+1)}{(z+1)(1-i)}$$

or, $w = i\left(\frac{1-z}{1+z}\right)$ which is the required transformation.

Now let $w = u + iv$, $z = x + iy$,

Therefore $$u + iv = i\left(\frac{1-x-iy}{1+x+iy}\right) = \frac{2y}{(1+x)^2+y^2} - i\frac{x^2+y^2-1}{(1+x)^2+y^2}$$

Thus $v = -i\frac{x^2+y^2-1}{(1+x)^2+y^2}$ therefore when $x^2+y^2=1$ i.e. $|z|=1$ ithen $v = 0$.

Module IV

(Partial Differential Equation (PDE) & Series Solution Of Ordinary Differential Equation (ODE))

Lecture-33

Basic Concepts of PDE

Origin of PDE:

With the knowledge of functions of several variables and the concept of a partial derivative, one can generalize the concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions and an additional variable for time.
Partial differential equations are ubiquitous in science, as they describe phenomena such as fluid flow, gravitational fields, and electromagnetic fields. They are important in fields such as aircraft simulation, computer graphics, and weather prediction. The central equations of general relativity and quantum mechanics are also partial differential equations.

Examples of some important PDEs:

(1) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ *Two-dimensional Laplace equation*

(4) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=f(x,y)$ *Two-dimensional Poisson equation*

Note that for PDEs one typically uses some other function letter such as u instead of y, which now quite often shows up as one of the variables involved in the multivariable function.

Order and Degree:

In general we can use the same terminology to describe PDEs as in the case of ODEs. For starters, we will call any equation involving one or more partial derivatives of a multivariable function a partial differential equation. The order of such an equation is the highest order partial derivative that shows up in the equation. In addition, the equation is called linear if it is of the first degree in the unknown function u, and its partial derivatives, u_x, u_{xx}, u_y, etc. (this means that the highest power of the function, u, and its derivatives is just equal to one in each term in the equation, and that only one of them appears in each term). If each term in the equation involves either u, or one of its partial derivatives, then the function is classified as homogeneous.

Take a look at the list of PDEs above. Try to classify each one using the terminology given above. Note that the f(x,y) function in the Poisson equation is just a function of the variables x and y, it has nothing to do with u(x,y).

Answers: All of these PDEs are second order, and are linear. All are also homogeneous except for the fourth one, the Poisson equation, as the f(x,y) term on the right hand side doesn't involve u or any of its derivatives.

The reason for defining the classifications linear and homogeneous for PDEs is to bring up the *principle of superposition*. This excellent principle (which also shows up in the study of linear homogeneous ODEs) is useful exactly whenever one considers solutions to linear homogeneous PDEs. The idea is that if one has two functions, u_1 and u_2 that satisfy a linear homogeneous differential equation, then since taking the derivative of a sum of functions is the same as taking the sum of their derivatives, then as long as the highest powers of derivatives involved in the equation are one (i.e., that it's linear), and that each term has a derivative in it (i.e. that it's homogeneous), then it's a straightforward exercise to see that the sum of u_1 and u_2 will also be a solution to the differential equation. In fact, so will any linear combination, au_1+bu_2, where a and b are constants.

For instance, the two functions $\cos(xy)$ and $\sin(xy)$ are both solutions for the first-order linear homogeneous PDE:

(5) $x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}=0$

It's a simple exercise to check that $\cos(xy)+\sin(xy)$ and $3\cos(xy)-2\sin(xy)$ are also solutions to the same PDE (as will be any linear combination of $\cos(xy)$ and $\sin(xy)$)

This principle is extremely important, as it enables us to build up particular solutions out of infinite families of solutions through the use of Fourier series.

Lecture-34

Solution PDEs:

Solving PDEs is considerably more difficult in general than solving ODEs, as the level of complexity involved can be great. For instance the following seemingly completely unrelated functions are all solutions to the two-dimensional Laplace equation:

(1) $\quad x^2 - y^2, \quad e^x \cos(y)$ and $\ln(x^2 + y^2)$

You should check to see that these are all in fact solutions to the Laplace equation by doing the same thing you would do for an ODE solution, namely, calculate $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$, substitute them into the PDE equation and see if the two sides of the equation are identical.

Now, there are certain types of PDEs for which finding the solutions is not too hard. For instance, consider the first-order PDE

(2) $$\frac{\partial u}{\partial x} = 3x^2 + xy^2$$

where u is assumed to be a two-variable function depending on x and y. How could you solve this PDE? Think about it, is there any reason that we couldn't just undo the partial derivative of u with respect to x by integrating with respect to x? No, so try it out! Here, note that we are given information about just one of the partial derivatives, so when we find a solution, there will be an unknown factor that's not necessarily just an arbitrary *constant*, but in fact is a completely arbitrary *function* depending on y.

To solve (2), then, integrate both sides of the equation with respect to x, as mentioned. Thus

(3) $$\int \frac{\partial u}{\partial x}\, dx = \int (3x^2 + xy^2)\, dx$$

so that $u(x, y) = x^3 + \frac{1}{2}x^2 y^2 + F$. What is F? Note that it could be any function such that when one takes its partial derivative with respect to x, the result is 0. This means that in the case of PDEs, the arbitrary constants that we ran into during the course of solving ODEs are now taking the form of whole functions. Here F, is in fact any function, $F(y)$, of y alone. To check that this is indeed a solution to the original PDE, it is easy enough to take the partial derivative of this $u(x, y)$ function and see that it indeed satisfies the PDE in (2).

Now consider a second-order PDE such as

(4) $$\frac{\partial^2 u}{\partial x \partial y} = 5x + y^2$$

where u is again a two-variable function depending on x and y. We can solve this PDE by integrating first with respect to x, to get to an intermediate PDE,

(5) $$\frac{\partial u}{\partial y} = \frac{5}{2}x^2 + xy^2 + F(y)$$

where $F(y)$ is a function of y alone. Now, integrating both sides with respect to y yields

(6) $$u(x, y) = \frac{5}{2}x^2 y + \frac{1}{3}xy^3 + F(y) + G(x)$$

where now $G(x)$ is a function of x alone (Note that we could have integrated with respect to y first, then x and we would have ended up with the same result). Thus, whereas in the ODE world, general solutions typically end up with as many arbitrary *constants* as the order of the original ODE, here in the PDE world, one typically ends up with as many arbitrary *functions* in the general solutions.

To end up with a specific solution, then, we will need to be given extra conditions that indicate what these arbitrary functions are. Thus the initial conditions for PDEs will typically involve knowing whole functions, not just constant values. We will also see that the initial conditions that appeared in specific ODE situations have slightly more involved analogs in the PDE world, namely there are often so-called *boundary* conditions as well as initial conditions to take into consideration.

Introduction to different methods of solution of PDEs:

Linear PDEs are generally solved, when possible, by decomposing the equation according to a set of basis functions, solving those individually and using superposition to find the solution corresponding to the boundary conditions. The method of separation of variables has many important particular applications.
There are no generally applicable methods to solve non-linear PDEs. Still, existence and uniqueness results (such as the Cauchy-Kovalevskaya theorem) are often possible, as are proofs of important qualitative and quantitative properties of solutions (getting these results is a major part of analysis).
Nevertheless, some techniques can be used for several types of equations. The h-principle is the most powerful method to solve underdetermined equations. The Riquier-Janet theory is an effective method for obtaining information about many analytic overdetermined systems.
The method of characteristics can be used in some very special cases to solve partial differential equations.
In some cases, a PDE can be solved via perturbation analysis in which the solution is considered to be a correction to an equation with a known solution. Alternatives are numerical analysis techniques ranging from finite difference schemes to multigrid, finite element and finite volume methods. Many interesting problems in science and engineering are solved in this way using **computers**, sometimes high performance supercomputers. However, most problems in science and engineering are tackled using scientific computing rather than numerical analysis, as usually it is not known whether the numerical methods used produce solutions close to the true ones.
Classification
Second-order partial differential equations, and systems of second-order PDEs, can usually be classified as parabolic, hyperbolic or elliptic. This classification gives an intuitive insight into the behaviour of the system itself. The general second-order PDE is of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots = 0,$$

which looks remarkably similar to the equation for a conic section:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

The reason B has a coefficient of 2 is due to the assumed commutativity of partial derivatives in the first case (recall that mixed derivatives which are continuous do not depend on the order of taking the partial derivatives in the different variables!) , and the commutativity of multiplication in the second. Just as one classifies conic sections into parabolic, hyperbolic, and elliptic based on the discriminant $B^2 - AC$, the same can be done for a second-order PDE.
$B^2 - AC < 0$: elliptic equations tend to smooth out any disturbances. A typical example is Laplace's equation. The motion of a fluid at sub-sonic speeds can be approximated with elliptic PDEs.
$B^2 - AC = 0$: parabolic equations tend to smooth out any pre-existing disturbances in the data. A typical example is the heat equation.
$B^2 - AC > 0$: hyperbolic equations tend to amplify any disturbances in the data. A typical example is the wave equation. The motion of a fluid at super-sonic speeds can be approximated with hyperbolic PDEs.
This method of classification can easily be extended to systems of partial differential equations by examining the eigenvalues of the coefficient matrix. In this situation, the classification scheme becomes:
Elliptic: The eigenvalues are all positive or all negative.
Parabolic : The eigenvalues are all positive or all negative, save one which is zero.
Hyperbolic : There is at least one negative and at least one positive eigenvalue, and none of the eigenvalues are zero.

This matches with positive-definite and negative-definite matrix analysis, of the sort that comes up during a discussion of maxima and minima. Moreover, using the concepts of positive-definiteness and negative-definiteness, it is possible to extend this classification to PDEs and systems of PDEs which are of order higher than 2 (as well as for systems of PDEs of 1st order).

Lecture-35

Solution by the method of Separation of Variables

One-dimensional Wave Equation:

There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner. This second solution is due to Jean Le Rond D'Alembert (an 18th century French mathematician), and is called D'Alembert's solution, as a result.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $\quad u(0,t)=0$ and $u(l,t)=0$ for all values of t

are the ***boundary conditions*** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

You might also note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the ***initial conditions*** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0)=0$ (a perfectly flat string) with initial velocity, $u_t(x,0)=0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t)=0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, x or t. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $\quad u(x,t)=F(x)G(t)$

where F, and G, are single variable functions of x and t respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $\quad \dfrac{\partial^2 u}{\partial x^2}=F''(x)G(t)$ and $\dfrac{\partial^2 u}{\partial t^2}=F(x)G''(t)$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $\quad \dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$

then we get

(5) $$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving F and its second derivative are on one side, and likewise the terms involving G and its derivative are on the other, then we get

(6) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Now we have an equality where the left-hand side just depends on the variable t, and the right-hand side just depends on x. Here comes the critical observation - how can two functions, one just depending on t, and one just on x, be equal for all possible values of t and x? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of t and x. Aha! Thus we have

(7) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

where k is a constant. First let's examine the possible cases for k.

Case One: k = 0

Suppose k equals 0. Then the equations in (7) can be rewritten as

(8) $G''(t) = 0 \cdot c^2 G(t) = 0$ and $F''(x) = 0 \cdot F(x) = 0$

yielding with very little effort two solution functions for F and G*:*

(9) $G(t) = at + b$ and $F(x) = px + r$

where *a,b, p* and *r*, are constants (note how easy it is to solve such simple ODEs versus trying to deal with two variables at once, hence the power of the separation of variables approach).

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us, namely that

(10) $u(0,t) = F(0)G(t) = 0$ and $u(l,t) = F(l)G(t) = 0$ for all values of t

Unless $G(t) = 0$ (which would then mean that $u(x,t) = 0$, giving us the very dull solution equivalent to a flat, unplucked string) then this implies that

(11) $F(0) = F(l) = 0$.

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x) = 0$. Sigh, then we still get that $u(x,t) = 0$, and we end up with the dull solution again, the only possible solution if we start with $k = 0$.

So, let's see what happens if…

Case Two: k > 0

So now if k is positive, then from equation (7) we again start with

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Try to solve these two ordinary differential equations. You are looking for functions whose second derivatives give back the original function, multiplied by a positive constant. Possible candidate solutions to consider include the exponential and sine and cosine functions. Of course, the sine and cosine functions don't work here, as their second derivatives are *negative* the original function, so we are left with the exponential functions.

Let's take a look at (13) more closely first, as we already know that the boundary conditions imply conditions specifically for $F(x)$, i.e. the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $\quad F(x) = Ae^{\omega x}$

where $\omega^2 = k$ and A is a constant. Since ω could be positive or negative, and since solutions to (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ODE, so that the superposition principle holds), then the general solution for (13) is

(14) $\quad F(x) = Ae^{\omega x} + Be^{-\omega x}$

where now A and B are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of A and B that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, i.e. the dull solution once more. Now we place all of our hope on the third and final possibility for *k*, namely…

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $\quad G''(t) = kc^2 G(t)$

and

(13) $\quad F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $\quad F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again A and B are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

(16) $\quad F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that B isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

(17) $\quad \omega l = n\pi \quad$ or $\quad \omega = \dfrac{n\pi}{l}$ (where *n* is an integer)

This means that there is an infinite set of solutions to consider (letting the constant *B* be equal to 1 for now), one for each possible integer *n*.

(18) $\quad F(x) = \sin\left(\dfrac{n\pi}{l}x\right)$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

(12) $\quad G''(t) = kc^2 G(t)$

where, again, we are working with *k,* a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with

$k = -\omega^2 = -\left(\dfrac{n\pi}{l}\right)^2$ for *n* some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

(19) $\quad G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t)$

where C and D are constants and $\lambda_n = c\sqrt{-k} = c\omega = \dfrac{cn\pi}{l}$, where n is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "n" to identify which value of n is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t) = F(x)G(t)$, and the result is

$$(20) \qquad u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$$

where the integer n that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and C and D are arbitrary constants.

At this point you should be in the habit of immediately checking solutions to differential equations. Is (20) really a solution for the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and does it actually satisfy the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t? Check this now – really, don't read any more until you're completely sure that this general solution works!

Lecture-36

One-dimensional Heat Equation:

We simplify our heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small, and constant, and insulated in such a way that the heat flow is just along the length of the bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, u, that keeps track of the temperature, just depends on x, the position along the bar, and t, time, and so the heat equation from the previous section becomes the so-called ***one-dimensional heat equation***:

$$(1) \qquad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

One of the interesting things to note at this point is how similar this PDE appears to the wave equation PDE. However, the resulting solution functions are remarkably different in nature. Remember that the solutions to the wave equation had to do with oscillations, dealing with vibrating strings and all that. Here the solutions to the heat equation deal with temperature flow, not oscillation, so that means the solution functions will likely look quite different. If you're familiar with the solution to Newton's heating and cooling differential equations, then you might expect to see some type of exponential decay function as part of the solution function.

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we're studying has a specific length, l, then we need to know the temperatures at the ends of the bars. These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation. To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time (you can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at 0 degrees). Thus we will be working with the same boundary conditions as before, namely

(2) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

Finally, to pick out a particular solution, we also need to know the initial starting temperature of the entire bar, namely we need to know the function $u(x,0)$. Interestingly, that's all we would need for an initial condition this time around (recall that to specify a particular solution in the wave equation we needed to know two initial conditions, $u(x,0)$ and $u_t(x,0)$).

The nice thing now is that since we have already solved a PDE, then we can try following the same basic approach as the one we used to solve the last PDE, namely separation of variables. With any luck, we will end up solving this new PDE. So, remembering back to what we did in that case, let's start by writing

(3) $u(x,t) = F(x)G(t)$

where *F*, and *G*, are single variable functions. Differentiating this equation for $u(x,t)$ with respect to each variable yields

(4) $\dfrac{\partial^2 u}{\partial x^2} = F''(x)G(t)$ and $\dfrac{\partial u}{\partial t} = F(x)G'(t)$

When we substitute these two equations back into the original heat equation

(5) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

we get

(6) $$\frac{\partial u}{\partial t} = F(x)G'(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

If we now separate the two functions *F* and *G* by dividing through both sides, then we get

(7) $$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Just as before, the left-hand side only depends on the variable *t*, and the right-hand side just depends on *x*. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, *k*:

(8) $$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

As before, let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions. From (8) we get that $F(x)$ has to satisfy

(9) $F''(x) - kF(x) = 0$

Just as before, one can consider the various cases with *k* being positive, zero, or negative. Just as before, to meet the boundary conditions, it turns out that *k* must in fact be negative (otherwise $F(x)$ ends up being identically equal to 0, and we end up with the trivial solution $u(x,t) = 0$). So skipping ahead a bit, let's assume we have figured out that *k* must be negative (you should check the other two cases just as before to see that what we've just written is true!). To indicate this, we write, as before, that $k = -\omega^2$, so that we now need to look for solutions to

(10) $F''(x) + \omega^2 F(x) = 0$

These solutions are just the same as before, namely the general solution is:

(11) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again A and B are constants and now we have $\omega = \sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$. These are equivalent to stating that $F(0) = F(l) = 0$. Substituting in 0 for x in (11) leads to

(12) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. As before, we check that B can't equal 0, otherwise $F(x) = 0$ which would then mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, the trivial solution, again. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. Again, the only way that this can happen is for ωl to be a multiple of π. This means that once again

(13) $\omega l = n\pi$ or $\omega = \dfrac{n\pi}{l}$ (where n is an integer)

and so

(14) $F(x) = \sin\left(\dfrac{n\pi}{l}x\right)$

where n is an integer. Next we solve for $G(t)$, using equation (8) again. So, rewriting (8), we see that this time

(15) $G'(t) + \lambda_n{}^2 G(t) = 0$

where $\lambda_n = \dfrac{cn\pi}{l}$, since we had originally written $k = -\omega^2$, and we just determined that $\omega = \dfrac{n\pi}{l}$ during the solution for $F(x)$. The general solution to this first order differential equation is just

(16) $G(t) = Ce^{-\lambda_n{}^2 t}$

So, now we can put it all together to find out that

(17) $u(x,t) = F(x)G(t) = C\sin\left(\dfrac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$

where n is an integer, C is an arbitrary constant, and $\lambda_n = \dfrac{cn\pi}{l}$. As is always the case, given a supposed solution to a differential equation, you should check to see that this indeed is a solution to the original heat equation, and that it satisfies the two boundary conditions we started with.

PDE Problems:

(1) Determine which of the following functions are solutions to the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(a) $u(x,y) = x^3 - 3xy^2$

(b) $u(x,y) = x^4 - 6x^2y^2 + y^4 + 12$

(c) $u(x,y) = \cos(x-y)e^{x-y}$

(d) $u(x,y) = \arctan(y/x)$

(e) $u(x,y) = 2002xy^2 + 1999x^2y$ (f) $u(x,y) = e^x(\sin(y) + 2\cos(y))$

(2) Determine which of the following functions are solutions to the one-dimensional wave equation (for a suitable value of the constant *c*). Also determine what *c* must equal in each case.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(a) $u(x,t) = \sin(x+2t) + \cos(x-2t)$ (b) $u(x,t) = \ln(xt)$

(c) $u(x,t) = 4x^3 + 12xt^2 + 24$ (d) $u(x,t) = \sin(100x)\sin(100t)$

(e) $u(x,t) = 2002xt^2 + 1001x^2t$ (f) $u(x,t) = x^2 + 4t^2$

(3) Find solutions $u(x,y)$ to each of the following PDEs by using the separation of variables technique.

(a) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$ (b) $\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$

(c) $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0$ (d) $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$

(e) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ (f) $\frac{\partial^2 u}{\partial x \partial y} - u = 0$

Lecture-37

Series Solution

Introduction

A **power series** is the sum of the infinite number of terms of the form

$$S = a_o + a_1(x - x_o) + a_2(x - x_o)^2 + \ldots = \sum_{m=0}^{\infty} a_m (x - x_o)^m$$

where $a_o,\ a_1,\ a_2,\ \ldots$ are constants, called the coefficients of the series. x_o is a constant, called the center of the series. A power series does not include terms with negative powers.

* The linear differential equations with constant coefficients always possess series solutions.

The homogeneous solution of the linear differential equation with constant coefficients

$$y'' + ay' + by = r(x)$$

will have one of the type:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
$$y = \left(A + Bx\right)e^{\lambda x}$$
$$y = (A\ cosax + B\ sinax)e^{\beta x}.$$

All of these solutions can be expanded in power series of x

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}$$

The power series form of y can be accepted as a solution provided that the differential equation is satisfied by it and the series is convergent. Some differential equations with variable coefficients possess series solutions.

Some properties of power series

The n-th partial sum of the series is

$S_n(x) = a_o + a_1(x - x_o) + a_2(x - x_o)^2 + \ldots + a_n(x - x_o)^n$

and the remainder is

$R_n(x) = a_{n+1}(x - x_o)^{n+1} + a_{n+2}(x - x_o)^{n+2} + \ldots$

A power series converges if $R_n \to 0$ as $n \to \infty$; otherwise, it diverges
There is usually an interval over which the power series converges with the center at $x = x_o$; that is, the series converges if

$$|x - x_o| < R$$

where R is called the **radius of convergence**. The radius of convergence can be obtained from

$$R = \lim\,(m \to \infty)\left|\frac{a_m}{a_{m+1}}\right|$$

EX: $e^x = 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^m}{m!} + \dfrac{x^{m+1}}{(m+1)!} + \ldots$

Radius of convergence

$$R = \lim\,(m \to \infty)\left|\frac{a_m}{a_{m+1}}\right| = \lim\,(m \to \infty)\left|\frac{\frac{1}{m!}}{\frac{1}{(m+1)!}}\right| = \lim\,(m \to \infty)\left|\frac{(m+1)!}{m!}\right|$$
$$= \lim\,(m \to \infty)\left|m + 1\right| \quad = \infty$$

A function y(x) is analytic at the point $x = x_o$ if it can be expressed as a power series $\sum_{m=0}^{\infty} a_m (x - x_o)^m$ with $R > 0$.

If the functions p(x), q(x), and r(x) in the differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

are analytic at the point $x = x_o$, the solution can be represented by a power series with a finite radius of convergence, that is,

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_o)^m \text{ with } R > 0$$

The point $x = x_o$ is called an ordinary (or regular) point.

EX: $y'' = e^x y$, every point $x \neq \infty$ is a regular point

$x^5 y'' = y$, every point x except for $x = 0$ and $x = \infty$ is a regular point

If p(x), q(x), or r(x) is not analytic at $x = x_o$, the point $x = x_o$ is said to be a singular point.

Regular singular point and irregular singular point

Consider a second order homogeneous linear equation

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{(s1)}$$

The point $x = x_o$ is called a regular singular point of (s1) if not both of p(x), q(x) are analytic but both $(x - x_o)p(x)$ and $(x - x_o)^2 q(x)$ are analytic in the neighborhood of x_o. The point $x = x_o$ is called an irregular singular point of (s1) if it is neither a regular point nor a regular singular point.

Ex: (a) $(x - 1)y'' = y$ has a regular singular point at 1

(b) $x^2 y'' + xy' = y$ has a regular singular point at 0

(c) $x^3 y'' = (x + 1)y$ has an irregular singular point at 0

If $x = x_o$ is a regular point of the differential equation (4.1-1) then the power series method can be applied. The general solution of Eq. (4.1-1) is $y = Ay_1(x) + By_2(x)$ where y_1 and y_2 are linearly independent series solutions $\sum_{m=0}^{\infty} a_m (x - x_o)^m$ which are analytic at $x = x_o$. The radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p(x) and q(x).

Lecture-38

Example: Find a power series solution of

$x(x+1)y' - (2x+1) = 0$.

Solution: The given equation is $x(x+1)y'-(2x+1)=0$.........(i)

Let $y=\sum_{n=0}^{\infty}c_n(x-x_0)^n$ be a solution of the equation (i). Thus $y'=\sum_{n=1}^{\infty}nc_n(x-x_0)^{n-1}$.

Putting these in equation (i) we get,

$$\sum_{n=1}^{\infty}\{(n-3)c_{n-1}+(n-1)c_n\}x^n-c_0x=0.$$

Equating the coefficients of x^n $(n=0,1,2,........)$ to zero we get

$c_0=0$, c_1 is arbitrary and $c_n=-\dfrac{n-3}{n-1}c_{n-1}$ for $n=2,3,4,.......$

Thus $c_2=c_1, c_n=0$ for $n\geq 3$.

LEGENDRE FUNCTION

The Legendre differential equation is given by

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)=0 \text{ ....(1)}$$

The parameter n which is a given integer (although it could be a real number) is called the order of the Legendre equation (1). The solution of the Legendre equation is known as the Legendre function of order n.

Assume that the power series solution of (1) as

$$y(x)=\sum_{m=0}^{\infty}a_m x^m \text{ ....(2)}$$

We get , $a_{k+2}=-\dfrac{(n-k)(n+k+1)}{(k+2)(k+1)}a_k$,   k=0,1,2,....

a_0, a_1 are arbitrary constants.

Substituting the coefficients in (1) we get the solution,

$$Y=a_0y_1(x)+a_1y_2(x) \text{ ....(3)}$$

Where, $y_1(x)=1-\dfrac{n(n+1)}{2!}x^2+\dfrac{(n-2)n(n+1)(n+3)}{4!}x^4$ ....(4)

$$y_2(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \text{ ....(5)}$$

Both the series (4) and (5) converge for $|x| < 1$. $y_1(x)$ and $y_2(x)$ are linearly independent. Thus y(x) is a general solution of (1) and is valid for $|x| < 1$ i.e. $-1 < x < 1$.

By the above method we can desire that the Legendre polynomial as,

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{(n-2m)} \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Where, M= $\frac{n}{2}$ or $\frac{(n-1)}{2}$ according as n is even or odd whichever is an integer.

In particular,

$$P_0(X) = 1, P_1(x) = x, P_2(x) = \frac{(3x^2-1)}{2}, P_3(x) = \frac{1}{2}(5x^3 - 3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Note : $P_n(1) = 1$.

Lecture-39

Rodrigue's Formula:

$$P_n(X) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2-1)^n$$

Proof: Let $v = (x^2-1)^n$.

Therefore,

$$\frac{dv}{dx} = n(x^2-1)^n.2x$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nx(x^2-1)^n$$

$$\Rightarrow (1-x^2)\frac{dv}{dx} + 2nxv = 0 \ldots\ldots\ldots(1)$$

Differentiating (1) ,(n+1) times by Leibnitz's rule,

$$\left\{(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} + {}^{n+1}c_1(-2x)\frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1}c_2(-2)\frac{d^n v}{dx^n}\right\} + 2n\left\{x\frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1}c_1.1.\frac{d^n v}{dx^n}\right\} = 0$$

$$\Rightarrow (1-x^2)\frac{d^{n+2}v}{dx^{n+2}} - 2x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n} = 0$$

Putting u= $\frac{d^n v}{dx^n}$, we get

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u = 0$$

Which is Legendre differential equation of order n and has a finite series solution $p_n(x)$. Thus U=c $p_n(x)$ ………..(2)

Here c is an arbitrary constant which is determined by equating the coefficients of x^n from (2) i.e.

$$cP_n(x) = u = \frac{d^n(x^2-1)^n}{dx^n}$$

Hence, $c\frac{(2n)!}{2^n(n!)^2} = 2n(2n-1)(2n-2)(2n-\overline{n-1}) = \frac{(2n)!}{n!}$, $\Rightarrow c = 2^n.n!$

Therefore, we get, $P_n(x) = \frac{1}{c}u = \frac{1}{(2n)n!}\frac{d^n(x^2-1)^n}{dx^n}$

Generating Function: The generating function for Legendre polynomial is $(1-2xt+t^2)^{-\frac{1}{2}}$ i.e.

$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$ ……….(1)

Result 1: $P_n(1) = 1$ for any n.

Proof: Putting x=1 , in (1) we get

$$(1-2t+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\Rightarrow \left\{(1-t^2)\right\}^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\Rightarrow (1-t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$1 + t + t^2 + \ldots\ldots + t^n + \ldots = \sum_{n=0}^{\infty} t^n P_n(1) = P_0(1) + tP_1(1) + t^2P_2(1) + \ldots + t^nP_n(1) + \ldots$$

Equating the coefficient of t^n, we get

$P_n(1) = 1$.

Result 2: $P_n(-1) = (-1)^n$ for any n.

Proof: Putting x=-1 in (1) we get,

$$(1+t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(-1)$$

Equating the coefficient of t^n, we get

$P_n(-1) = (-1)^n$

Result 3: $P_n(0) = \begin{cases} o, n = \text{odd} \\ (-1)^{\frac{n}{2}} \dfrac{1.3.5.....(n-1)}{2.4.6....n}, n = \text{even} \end{cases}$

Proof: $\sum_{n=0}^{\infty} P_n(0)t^n = (1+t^2)^{-\frac{1}{2}}$

$$= 1 - \frac{1}{2}t^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1).......(-\frac{1}{2}-\overline{n-1})}{1.2.3.4.5....n}t^{2n} + ...$$

$$= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}t^4 + ... + \frac{(-1)^n 1.3.5.....(2n-1)}{2.4.6....2n}t^{2n}$$

From both sides equating the coefficient of equal power of t^{2n} and t^{2n+1} we get,

$P_{2n+1}(0) = 0$

$P_{2n}(0) = (-1)^n \dfrac{(-1)^n 1.3.5.....(2n-1)}{2.4.6....2n}$

Hence we get the result by replacing 2n by n.

Lecture-40

Recurrence Relations:

1. $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$

Proof: From Rodrigue's formula

$P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2-1)^n$

Differentiating with respect to x,

$$P'_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left\{n(x^2-1)^{n-1}.2x\right\}$$

$$= \frac{1}{2^n (n-1)!} \frac{d^n}{dx^n}\left\{x(x^2-1)^{n-1}\right\}$$

By Leibnitz's rule we get,

$$P'_n(x) = \frac{1}{2^{n-1}(n-1)!}\left[x\frac{d^n}{dx^n}x^2-1)^{n-1} + {}^nc_1 1.\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right]$$

$$= x\frac{d}{dx}\left\{\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right\} + n\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$$

$$= xP'_{n-1}(x) + nP_{n-1}(x).$$

2. $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

Proof: $P'_{n+1}(x) = \frac{d}{dx}\left\{\frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}\right\}$

$$= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}\left\{(n+1)(x^2-1)^n.2x\right\}$$

$$= \frac{1}{2^n(n)!}\frac{d^{n+1}}{dx^{n+1}}\left\{x(x^2-1)^n\right\}$$

$$= \frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left\{x.n(x^2-1)^{n-1}.2x + (x^2-1)^n\right\}$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}\left\{x^2(x^2-1)^{n-1}\right\} + P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}\left\{((x^2-1)+1)(x^2-1)^{n-1}\right\} + P_n(x)$$

$$= 2n P_n(x) + P'_{n-1}(x) + P_n(x)$$

$\Rightarrow P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

3. $xP'_n(x) = nP_n(x) + P'_{n-1}(x)$

Proof: From (1) we get,

$P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$

Replacing n by (n+1) we get,

$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$

From (2) we get,

$P'_{n+1}(x) = P'_{n-1}(x) + (2n+1)P_n(x)$

Therefore,

$xP'_n(x) + (n+1)P_n(x) = P'_{n-1}(x) + (2n+1)P_n(x)$

$\Rightarrow xP'_n(x) = nP_n(x) + P'_{n-1}(x)$

4. $(1-x^2)P'_{n-1}(x) = n(xP_{n-1}(x) - P_n(x))$

Proof: $(1) \Rightarrow P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$ …….(1)

$$(3) \Rightarrow xP'_n(x) = nP_n(x) + P'_{n-1}(x) \text{ ....(2)}$$

By (1) $\times x$ - (2) we get,

$$0 = \left(x^2 - 1\right)P'_{n-1}(x) + n\left[xP_{n-1}(x) - P_n(x)\right]$$

$$\Rightarrow (1 - x^2)P'_{n-1}(x) = n\left[xP_{n-1}(x) - P_n(x)\right]$$

5. $\left(x^2 - 1\right)P'_n(x) = n(xP_n(x) - P_{n-1}(x))$

Proof: $(1) \Rightarrow P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x) \text{ ......(i)}$

$$(3) \Rightarrow xP'_n(x) = P'_{n-1}(x) + nP_n(x) \text{ ....(ii)}$$

By (ii)$\times x$ -(i) we get,

$$\left(x^2 - 1\right)P'_n(x) = n(xP_n(x) - P_{n-1}(x))$$

6. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Proof: $(4) \Rightarrow (1 - x^2)P'_{n-1}(x) = n\left[xP_{n-1}(x) - P_n(x)\right]$

Replacing n by (n+1) we get,

$$(n+1)P_{n+1}(x) = (n+1)xP_n(x) - (1 - x^2)P'_n(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (n+1)xP_n(x) + n(xP_n(x) - P_{n-1}(x)) \text{ (from (5))}$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Lecture-41

Orthogonality of Legendre Polynomials

If $P_n(x)$ is the Legendre polynomial of order n, then

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0, m \neq n \\ \dfrac{2}{2n+1}, m = n \end{cases}$$

Proof: From Legendre differential equation, we get

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = -n(n+1)y$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} = -n(n+1)y \text{ ....(1)}$$

Since $P_n(x)$ is the solution of equation (1), we get

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} = -n(n+1)y \text{ ....(2)}$$

And also, $\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} = -m(m+1)y$ ....(3)

By (2)X P_m -(3)X P_n, we get

$$P_m\frac{d}{dx}\left\{(1-x^2)P'_n\right\} - P_n\frac{d}{dx}\left\{(1-x^2)P'_m\right\} = -\left\{n(n+1)-m(m+1)\right\}P_mP_n$$

$$\Rightarrow P_m\frac{d}{dx}\left\{\left(1-x^2\right)P'_n\right\} + (1-x^2)P'_n\frac{d}{dx}P_m - P_n\frac{d}{dx}\left\{\left(1-x^2\right)P'_m\right\} - \left(1-x^2\right)P'_m\frac{d}{dx}P_n = -\left\{n^2+n-m^2-m\right\}P_mP_n$$

$$\Rightarrow \frac{d}{dx}\left\{\left(1-x^2\right)P'_nP_m\right\} - \frac{d}{dx}\left\{\left(1-x^2\right)P'_mP_n\right\} = -\left\{(m+n)(n-m)+(n-m)\right\}P_mP_n$$

$$\Rightarrow \frac{d}{dx}\left\{\left(1-x^2\right)(P'_nP_m - P'_mP_n\right\} = -(n-m)(n+m+1)P_mP_n$$

$$\Rightarrow d\left\{\left(1-x^2\right)(P'_nP_m - P'_mP_n\right\} = -(n-m)(n+m+1)P_mP_n\,dx$$

Integrating between the limits -1 to 1 we get,

$$\left[\left(1-x^2\right)(P'_nP_m - P'_mP_n\right]_{-1}^{1} = -(n-m)(n+m+1)\int_{-1}^{1}P_mP_n$$

$$\Rightarrow 0 = -(n-m)(n+m+1)\int_{-1}^{1}P_m(x)P_n(x)dx$$

$$\Rightarrow \int_{-1}^{1}P_m(x)P_n(x)dx = 0, \text{if } m \neq n$$

Again,

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}t^nP_n(x)$$

Squaring both sides we get,

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty}t^{2n}\left\{P_n(x)\right\}^2 + 2\sum_{\substack{m,n=0\\m\neq n}}^{\infty}t^{m+n}P_m(x)P_n(x)$$

Integrating between the limits -1 to 1 we get,

$$\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}\left\{P_n(x)\right\}^2dx + 2\sum_{\substack{m,n=0\\m\neq n}}^{\infty}t^{m+n}\int_{-1}^{1}P_m(x)P_n(x)dx = \int_{-1}^{1}\frac{dx}{1-2xt+t^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} \{P_n(x)\}^2 dx = -\frac{1}{2t}\left[\log(1-2xt+t^2)\right]_{-1}^{1}$$

$$= -\frac{1}{2t}\left[\log(1-2t+t^2) - \log(1-2t+t^2)\right]$$

$$= -\frac{1}{2t}\left[\log(1-t)^2 - \log(1+t)^2\right]$$

$$= -\frac{1}{2t}\log(\frac{1-t}{1+t})^2$$

$$= \frac{1}{t}\log(\frac{1+t}{1-t})$$

$$= \frac{1}{t}2\left\{t + \frac{t^3}{3} + \frac{t^5}{5} + ...\right\}$$

$$= 2\left\{1 + \frac{t^2}{3} + \frac{t^4}{5} + ...\right\}$$

$$\Rightarrow \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} \{P_n(x)\}^2 dx = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}$$

Equating the co-efficient of t^{2n} we get

$$\int_{-1}^{1} \{P_n(x)\}^2 dx = \frac{2}{2n+1}$$

i.e. $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}$, if m=n

Lecture-42

Bessel's Function

TheBessel's D.E is given by

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - p^2\right)y = 0 \quad ....(1)$$

Here ,p,which is a given constant (may not be integer)is known as the order of the Bessel's equation.
Using Frobeniusmethod , assuming p is real and non- negative, let the solution be

$$y = x^r\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r} [a_0 \neq 0]$$

The initial equation becomes r^2-p^2=0. Whose roots are r=$\pm$p.
The two solutions of equation (1) becomes

$$y_1(x) = J_p(x) = x^p\sum_{m=o}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+p}\underline{|m}\Gamma(m+p+1)} \text{ and}$$

$$y_2(x) = J_{-p}(x) = x^{-p}\sum_{m=o}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m-p}\underline{|m}\Gamma(m-p+1)}$$

The general solution of (1)will be

$y = C_1 J_p(x) + C_2 J_{-p}(x)$, provided p is not an integer.

Linear dependence of Bessel's Function:

Let p=n be an integer. Then

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \lfloor m \Gamma(m+n+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} \lfloor m \lfloor (m+n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Replacing n by –n we get,

$$J_{-n}(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n} \lfloor m \Gamma(m-n+1)} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

When (m-n+1) ≤ 0 i.e. m $\le$ (n-1), the gamma function of zero or negative integers become infinite. Thus For m=0 to (n-1), the coefficient in (3) becomes zero. So m starts at n.

Hence, $$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} \lfloor m \lfloor (m-n)} \quad \text{[put s=m-n]}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+n} x^{2s+n}}{2^{2s+n} \lfloor s \lfloor (s+n)}$$

$$= (-1)^n J_n(x)$$

i.e. $J_{-n}(x) = (-1)^n J_n(x)$

Lecture-43

Recurrence relations:

1. $\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$

Proof: $$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p}}{2^{2m+p} \lfloor m \Gamma(m+p+1)}$$

$$x^p J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2p}}{2^{2m+p} \lfloor m \lfloor (m+p)}$$

$$\frac{d}{dx}\{x^p J_p(x)\} = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2p) x^{2m+2p-1}}{2^{2m+p} \lfloor m (m+p) \Gamma(m+p)}$$

$$= x^p \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+p) x^{2m+p-1}}{2^{2m+p} \lfloor m (m+p) \Gamma(m+(p-1)+1)}$$

$$= x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+p-1}}{2^{2m+(p-1)} \lfloor m \Gamma(m+(p-1)+1)}$$

$$= x^p J_{p-1}(x)$$

2. $\dfrac{d}{dx}\{x^{-p}J_p(x)\} = -x^{-p}J_{p+1}(x)$

Proof: $\dfrac{d}{dx}\{x^{-p}J_p(x)\} = \dfrac{d}{dx}\{\sum_{m=0}^{\infty}\dfrac{(-1)^m x^{2m}}{2^{2m+p}\lfloor m\,\Gamma(m+p+1)}$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m 2mx^{2m-1}}{2^{2m+p}\lfloor m\,\Gamma(m+p+1)}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m+p-1}\lfloor m-1\,\Gamma(m+p+1)}$$

$$= \sum_{s=0}^{\infty}\frac{(-1)^{s+1} x^{2(s+1)-1}}{2^{2(s+1)+p-1}\lfloor s\,\Gamma(s+1+p+1)} \qquad \text{[put m-1=s]}$$

$$= -x^{p}\sum_{s=0}^{\infty}\frac{(-1)^{s} x^{2s+p+1}}{2^{2s+(p+1)}\lfloor s\,\Gamma(s+(p+1)+1)}$$

$$= -x^{-p}J_{p+1}(x)$$

3. $\dfrac{d}{dx}\{J_p(x)\} = J_{p-1}(x) - \dfrac{p}{x}J_p(x)$

Or $xJ_p^{'}(x) = xJ_{p-1}(x) - pJ_p(x)$.

Proof: We know that

$$\frac{d}{dx}\{x^{p}J_p(x)\} = x^{p}J_{p-1}(x)$$

$$x^{p}\frac{d}{dx}\{J_p(x)\} + px^{p-1}J_p(x) = x^{p}J_{p-1}(x)$$

$$\frac{d}{dx}\{J_p(x)\} = J_{p-1}(x) - \frac{p}{x}J_p(x)$$

4. $J_p^{'}(x) = \dfrac{p}{x}J_p(x) - J_{p+1}(x)$

Proof: $\dfrac{d}{dx}\{x^{-p}J_p(x)\} = -x^{-p}J_{p+1}(x)$

$$x^{-p}\frac{d}{dx}\{J_p(x)\} - px^{-p-1}J_p(x) = -x^{p}J_{p+1}(x)$$

$$J_p^{'}(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$$

5. $J_p^{'}(x) = \dfrac{1}{2}\{J_{p-1}(x) - J_{p+1}(x)\}$

Proof: Adding (3)& (4) we get.

6. $J_{p-1}(x) - J_{p+1}(x) = \dfrac{2p}{x}J_p(x)$

Proof: Subtracting (4) from (3) we get.

Lecture-44

Elementary Bessel's Function

Result 1: $J_{\frac{1}{2}}(x)=\sqrt{\dfrac{2}{\pi x}}\sin x$

Proof: $J_p(x)=x^p\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+p}\lfloor\underline{m}\,\Gamma(m+p+1)}$

Putting p=1/2 we get,

$$J_{\frac{1}{2}}(x)=x^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}}\lfloor\underline{m}\,\Gamma(m+\frac{3}{2})}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, $\Gamma(m+\frac{3}{2})=(m+\frac{1}{2})(m-\frac{1}{2})(m-\frac{3}{2})\ldots\ldots\ldots\frac{3}{2}.\frac{1}{2}.\Gamma(\frac{1}{2})$

$$=\frac{(2m+1)(2m-1)(2m-3)\ldots\ldots3.1}{2^{m+1}}\sqrt{\pi}$$

Also, $2^{2m+1}.\lfloor\underline{m}=2^{m+1}.2^m.\lfloor\underline{m}$

$$=2^{m+1}.2^m.m.(m-1)\ldots\ldots2.1$$
$$=2^{m+1}.2m.2(m-1)\ldots\ldots4.2$$

Thus, $2^{2m+1}.\lfloor\underline{m}\,\Gamma(m+\frac{3}{2})=\{2^{m+1}.2m.2(m-1)\ldots4.2\}\dfrac{(2m+1)(2m-1)(2m-3)\ldots\ldots3.1}{2^{m+1}}\sqrt{\pi}$

$$=\{2m.(2m-2).(2m-4)\ldots.4.2\}\,\{(2m+1)(2m-1)(2m-3)\ldots\ldots\ldots3.1\}\sqrt{\pi}$$
$$=\lfloor\underline{(2m+1)}\sqrt{\pi}$$

Then from (1) we get,

$$J_{\frac{1}{2}}(x)=x^{-\frac{1}{2}}.x\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+1}.2^{-\frac{1}{2}}\lfloor\underline{m}\,\Gamma(m+\frac{3}{2})}$$

$$=(\frac{2}{x})^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}\lfloor\underline{m}\,\Gamma(m+\frac{3}{2})}$$

$$=(\frac{2}{x})^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{\lfloor\underline{(2m+1)}\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

Result 2: $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Proof: We have from the previous result,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

Again, $\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$

$$\frac{d}{dx}\{x^{\frac{1}{2}} J_{\frac{1}{2}}(x)\} = x^{\frac{1}{2}} J_{-\frac{1}{2}}(x)$$

$$\frac{d}{dx}\{x^{\frac{1}{2}} \sqrt{\frac{2}{\pi x}} \sin x\} = x^{\frac{1}{2}} J_{-\frac{1}{2}}(x)$$

$$J_{-\frac{1}{2}}(x) = x^{-\frac{1}{2}} \frac{d}{dx}\{\sqrt{\frac{2}{\pi}} \sin x\}$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$